2019

An exploration of anti-van der Waerden numbers

Alex Schulte
Iowa State University

An exploration of anti-van der Waerden numbers

by

Alex Schulte

A dissertation submitted to the graduate faculty

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Major: Mathematics

Program of Study Committee:
Michael Young , Major Professor
Steve Butler
Jonas Hartwig
Leslie Hogben
Bernard Lidický

The student author, whose presentation of the scholarship herein was approved by the program of study committee, is solely responsible for the content of this dissertation. The Graduate College will ensure this dissertation is globally accessible and will not permit alterations after a degree is conferred.

Iowa State University

Ames, Iowa

2019

DEDICATION

I would like to dedicate this thesis to my wife Quinn and to my sister Lacie without whose support I would not have been able to complete this work. I would also like to thank my mother Vickie for never giving up on me and my father Donald for always being willing to lend a helping hand. I would also like to thank my friends and family for their loving guidance throughout my graduate education.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

ACKNOWLEDGMENTS

I would like to take this opportunity to express my thanks to those who helped me with various aspects of conducting research and the writing of this thesis. First and foremost, Dr. Michael Young for his guidance, patience and support throughout this research and the writing of this thesis. His insights and words of encouragement have often inspired me and renewed my hopes for completing my graduate education. I would also like to thank my committee members for their efforts and contributions to this work: Dr. Steve Butler, Dr. Jonas Hartwig, Dr. Leslie Hogben, and Dr. Bernard Lidický. I would also like to thank the entire discrete math group here at Iowa State University for supporting me and providing me with a multitude of opportunities. I would like to thank Dr. Nathan Warnberg at UW La Crosse for helping mentor me throughout our research collaborations. I would additionally like to thank Dr. Zhanar Berikkyzy, Dr. Ruth Haas, Hunter Rehm, Chase Weida, and Matthew Riley for conducting research with me on anti-van der Waerden numbers. Finally, I would like to thank Dr. Robert Lazar and Kassie Atwood for proof reading and helping edit this dissertation.

ABSTRACT

In this paper results of the anti-van der Waerden number of various mathematical objects are discussed. The *anti-van der Waerden number* of a mathematical object G, denoted by $\mathrm{aw}(G,k)$, is the smallest r such that every exact r-coloring of G contains a rainbow k-term arithmetic progression. In this paper, results on the anti-van der Waerden number of the integers, groups such as the integers modulo n, and graphs are given. A connection between the Ramsey number of paths and the anti-van der Waerden number of graphs is established. The anti-van der Waerden number of $[m] \times [n]$ is explored. Finally, connections between anti-van der Waerden numbers, rainbow numbers, and anti-Schur numbers are discussed.

Keywords anti-van der Waerden number; Ramsey number; rainbow; k-term arithmetic progression.

CHAPTER 1. INTRODUCTION

In this dissertation an overview of the anti-van der Waerden number will be given. Many directions are yet unexplored in this area of mathematics and there are a multitude of directions that this research can be taken in the future. This chapter will give historical reference to the problem as well as give definitions and basic results that will be used throughout the dissertation.

In general, the anti-van der Waerden number can be thought of as the minimum number of colors used on a set S, to ensure there exists a subset F such that each element of F is uniquely colored. In this dissertation the subset F is primarily thought of as a k-term arithmetic progression. Various settings for S are considered such as $[n]$, abelian groups such as $\mathbb{Z}_n$, non-abelian groups, and graphs.

Ramsey theory focuses on the idea that any large enough set contains a subset with some sort of order. This began in graph theory by determining the smallest n for which a 2-coloring of the edges of K_n has a monochromatic K_k or K_ℓ in a particular color. The study of Ramsey theory, searching for monochromatic subsets, led to a similar study named anti-Ramsey theory where subsets that are rainbow are found. That is, what is the smallest n such that a rainbow K_k exists with conditions on the number of colors. A key result of Ramsey theory is van der Waerdens theorem which proves that for every k and r there exists an integer n such that a r-coloring of $[n]$ contains a monochromatic k-term arithmetic progression. The minimum n that guarantees a rainbow k-term arithmetic progression is named van der Waerden's number. Again, this investigation of monochromatic k-term arithmetic progressions led to the study of rainbow k-term arithmetic progressions and the concept of anti-van der Waerden numbers.

The historical context and definitions are given in this chapter. In Chapter 2, results for $[n]$, $\mathbb{Z}_n$, abelian groups, and non-abelian groups are given for 3-term arithmetic progressions. In Chapter 3, the anti-van der Waerden number of graphs is explored. In Chapter 4, the focus moves from

3-term arithmetic progressions to k-term arithmetic progressions with $k \geq 4$. In Chapter 5, the anti-van der Waerden number of $[m] \times [n]$, weighted graphs, directed graphs, and other variations on anti-van der Waerden numbers such as rainbow numbers and anti-Schur numbers are discussed. In Chapter 6, closing remarks and a list of potential projects that could be explored in the future are discussed.

1.1 Historical Context

Problems involving counting and the existence of rainbow arithmetic progressions have been well-studied. The main results of Axenovich and Fon-Der-Flaass [1] and Axenovich and Martin [2] deal with the existence of 3-APs in colorings that have uniformly sized color classes. Fox, Jungić, Mahdian, Nešetril, and Radoičić also studied anti-Ramsey results of arithmetic progressions in [16]. In particular, they showed that every 3-coloring of $[n]$ for which each color class has density more than $\frac{1}{6}$, contains a rainbow 3-AP. The anti-van der Waerden number was first defined by Uherka in [28]. Many results on arithmetic progressions of $[n]$ and the cyclic groups $\mathbb{Z}_n$ were considered by Butler, Erickson, Hogben, Hogenson, Kramer, Kramer, Lin, Martin, Stolee, Warnberg, and Young in [7]. Moreover, Butler et al. determined the anti-van der Waerden number of $\mathbb{Z}_n$ for all n when $k = 3$. A function $f(n)$ was established by Berikkyzy, Schulte, and Young in [4] such that $\operatorname{aw}([n], 3) = f(n)$ for all $n \in \mathbb{N}$. Young extended colorings and 3-term arithmetic progressions to groups in [31].These were then extended to graphs by Schulte, Warnberg, and Young in [27]. Schulte et al. in [27] were inspired to investigate the anti-van der Waerden number of graphs by extending results on the anti-van der Waerden number of $[n]$ and $\mathbb{Z}_n$ to paths and cycles, respectively. In particular, they noticed that the set of arithmetic progressions on $[n]$ is isomorphic to the set of non-degenerate arithmetic progressions on P_n. Similarly, the set of arithmetic progressions on $\mathbb{Z}_n$ is isomorphic to the set of non-degenerate arithmetic progressions on C_n. Therefore, considering the anti-van der Waerden number of $[n]$ or $\mathbb{Z}_n$ is equivalent to studying the anti-van der Waerden number of paths or cycles, respectively. Rehm, Schulte, and Warnberg found an upper bound for the anti-van der Waerden number of graph products in [22], proving a conjecture made in [27]. Rehm et al. also determined the anti-van der Waerden number for the cartesian product of any two paths.

Ramsey theory is a popular and thriving area of discrete mathematics. The theory originally began in mathematical logic, but Ramsey theory is rarely studied in that context anymore. Ramsey theory has formed connections to many areas of mathematics including algebra, geometry, set theory, ergodic theory, number theory, topology, graph theory, and combinatorics [24]. Informally, Ramsey's theorem states: *Any structure will necessarily contain an orderly substructure* [28]. That is, when a system is large enough, even though it may look very complex, there is a structure within the system that has order. Besides Frank Ramsey, many other noteworthy figures, including but not limited to, Bartel van der Waerden, Issai Schur, and Richard Rado, facilitated the advancement of Ramsey theory during its early years [17]. Paul Erdős, who has been called *the father of modern Ramsey theory*, contributed to Ramsey theory's revival during the mid nineteen hundreds and is credited with the popularization of Ramsey theory amongst the contemporary generation of mathematicians [17].

The concept of the anti-van der Waerden number comes from the study of Ramsey and anti-Ramsey problems. The main result pertaining to Ramsey theory on the integers is van der Waerden's theorem (Theorem 1) [29]. Bartel van der Waerden was a gifted Dutch mathematician who was primarily interested in algebraic geometry until one of his students presented him with a problem concerning arithmetic progressions that had been proposed a number of years beforehand, it immediately caught his interest [13]. Van der Waerden's theorem was published in *Beweis einer Baudetschen Vermutung* in 1927 [29], one year before Ramsey's theorem was published.

1.2 Background and Definitions

In this section, definitions and results that will be relevant throughout this dissertation are given. The section begins with initial definitions and the formal statement of Van der Waerden's Theorem.

1.2.1 Van der Waerden Numbers

An *arithmetic progression* has an initial value a, and an increasing value, $d > 0$. A $k-$term arithmetic progression, henceforth referred to as a k-AP, is k terms $a, a+d, a+2d, \ldots a+(k-1)d$. For the purposes of this dissertation, a k-AP is referred to as a set of the form $\{a, a+d, a+2d, \ldots, a+(k-1)d\}$ where the order terms appear in the set coincide with the order of the arithmetic progression.

An *r-coloring* of a set S is a function from $c : S \to [r]$ where $\{1, \ldots, r\}$ is a set of r colors. An *exact r-coloring* of a set S is a surjective function $c : S \to [r]$. A set $S' \subseteq S$ is *monochromatic* under coloring c if for any pair $s_1, s_2 \in S'$, $c(s_1) = c(s_2)$. A set $S' \subseteq S$ is *rainbow* under coloring c if for any pair $s_1, s_2 \in S'$, $c(s_1) \neq c(s_2)$ when $s_1 \neq s_2$.

Let c be a coloring of a set S. The *color class* of a color α, denoted $c_\alpha(S)$, is a set of items that are colored α. Note for an exact r-coloring, the cardinality of each color class is at least 1. A *unitary coloring* is a coloring for which the cardinality of a color class is exactly 1 for at least one color.

Theorem 1 (Van der Waerden's Theorem)**.** *[29] For any given positive integers r and k, there exists a positive integer n such that every r-coloring on $[n]$ contains a monochromatic k-AP.*

To this end, define the *van der Waerden number*, denoted $\mathrm{w}(r, k)$, to be the least positive integer w such that every r-coloring of the integers $[w]$ contains a monochromatic k-AP.

Table 1.1 gives some known van der Waerden values [17].

The van der Waerden number is defined in [29] and a proof of van der Waerden's theorem is given in [17].

1.2.2 Anti-van der Waerden Numbers

In Ramsey theory, substructures that are monochromatic are studied whereas in anti-Ramsey theory, substructures that are rainbow are identified. An analogous relationship exists between the van der Waerden number and the anti-van der Waerden number. The van der Waerden number investigates the minimum n such that every exact r-coloring of $[n]$ has a monochromatic k-AP,

Table 1.1 Known values for $\mathrm{w}(r,k)$

r	k	$\mathrm{w}(r,k)$
2	3	9
2	4	35
2	5	178
2	6	1132
3	3	27
4	3	76
1	k	k
r	1	1
r	2	$r+1$

and the anti-van der Waerden number identifies the minimum number of colors r such that every exact r-coloring of $[n]$ contains a rainbow k-AP.

For positive integers n and k, the *anti-van der Waerden number*, denoted $\mathrm{aw}([n],k)$, is the least positive integer r such that every exact r-coloring of the integers from 1 to n contains a rainbow k-AP. If no such coloring exists, then $\mathrm{aw}([n],k) = n+1$. Note that $k \leq \mathrm{aw}([n],k) \leq n+1$.

In the investigation of the anti-van der Waerden number, Butler et al. noticed that many extremal colorings were unitary colorings [7]. This led to the definition of the unitary anti-van der Waerden number defined by Berikkyzy et al. in [4].

The smallest r such that every exact unitary r-coloring of $[n]$ contains a rainbow k-AP is the *unitary anti-van der Waerden number*, denoted $\mathrm{aw}_u([n],k)$. Similar to the anti-van der Waerden number, $\mathrm{aw}_u([n],k) = n+1$ if $[n]$ has no k-AP.

A set of consecutive integers I in $[n]$ is called an *interval* and $\ell(I)$ is the number of integers in I. Given a coloring c of some finite nonempty subset S of $[n]$, a *color class* of a color i under c in S is denoted $c_i(S) := \{x \in S : c(x) = i\}$. A coloring c of $[n]$ is *special* if $n = 7q+1$ for some positive integer q, $c(1)$ and $c(n)$ are both uniquely colored, and there are two colors α and β such that $c_\alpha([n]) = \{q+1, 2q+1, 4q+1\}$ and $c_\beta([n]) = \{3q+1, 5q+1, 6q+1\}$.

The anti-van der Waerden number can be defined by the same definition for a group G where the k-AP is $\{a, a+d, \ldots, a+(k-1)d\}$ where $+$ denotes the group operation. In the case of a non-abelian group, G, with operation $+$ and $a, d \in G$, a *right k-term arithmetic progression* of G is a set of the form $\{a, a+d, \ldots, a+(k-1)d\}$ and a *left k-term arithmetic progression* of G is a set of the form $\{a, d+a, \ldots, (k-1)d+a\}$.

1.2.3 Graphs

A *graph*, G, is a set of vertices, $V(G)$, and a set of edges, $E(G)$, and will be denoted as $G = (V, E)$. The edge set E is a set of pairs of vertices that indicate the two vertices are connected. Thus, if there is an edge connecting vertices u and v, then $\{u, v\}$ is an edge or uv is an edge for short. For the purposes of this dissertation all graphs are simple (loop free, no multiple edges). The *order* of G is the cardinality of $V(G)$, denoted either $|V(G)|$ or $|G|$ for short. The *size* of G is the cardinality of $E(G)$, denoted $|E(G)|$. Graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An *induced subgraph* H of G is one formed by deleting vertices of G and their incident edges. The induced subgraph with vertex set U is denoted $G[U]$.

Vertices u and v are *adjacent* in G if $uv \in E(G)$. A vertex $v \in V(G)$ and an edge $e \in E(G)$ are *incident* in G if $e = uv$, for some $u \in V(G)$.

Let $v \in V(G)$. A vertex u is a *neighbor* of v if $uv \in E(G)$. The *neighborhood* of v, denoted by $\mathrm{N}(v)$ is the set of neighbors of v. The *degree* of v, $\deg(v)$ is the number of edges incident to v. Note: $\deg(v) = |\mathrm{N}(v)|$. A vertex with degree equal to 0 is called an *isolated vertex*. A vertex with degree equal to 1 is called a *leaf*.

The *minimum degree* of G is $\delta(G) := \min\{\deg(v) | v \in V(G)\}$ and the *maximum degree* of G is $\Delta(G) := \max\{\deg(v) | v \in V(G)\}$. A graph G is *regular* if $\Delta(G) = \delta(G)$ and G is *cubic* if $\Delta(G) = 3 = \delta(G)$.

A *complete graph* on n vertices, written K_n, is a graph such that there is an edge between every pair of vertices. An *empty graph* or *independent graph* on n vertices is a graph on n vertices with no edges. A *path*, denoted P_n, is the graph where $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-2}v_{n-1}, v_{n-1}v_n\}$. A *cycle*, denoted C_n, is the graph where $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$.

A graph G is *r-partite* if $V(G)$ can be partitioned into r sets, $V_1, V_2, \ldots, V_r$ such that $G[V_i]$ has no edges for all $1 \leq i \leq r$. A *bipartite* graph is a 2-partite graph. A *complete bipartite* graph is a bipartite graph with all possible edges between the two parts and is denoted $K_{m,n}$ where m and n are the number of vertices in the bipartite sets.

The *distance* between vertex u and v in graph G is denoted $\mathrm{d}_G(u,v)$, $\mathrm{d}(u,v)$ will be used when the context is clear. A subgraph H of G is *isometric* if for all $u, v \in V(H)$ $\mathrm{d}_H(u,v) = \mathrm{d}_G(u,v)$.

A graph G is *connected* if for each pair of vertices $u, v \in V(G)$ there exists a path from u to v in G. A graph that is not connected is called *disconnected*. A *connected component* of G is a maximal connected subgraph of G.

A vertex v in graph G is *dominating* if v is adjacent to each other vertex of G. The *eccentricity* of vertex v in graph G is $\mathrm{ecc}(v) = \max\{\mathrm{d}(v,u) \mid u \in V(G)\}$. The *radius* of graph G is $\mathrm{rad}(G) = \min\{\mathrm{ecc}(v) \mid v \in V(G)\}$ and the *diameter* is $\mathrm{diam}(G) = \max\{\mathrm{ecc}(v) \mid v \in V(G)\}$. A vertex v is *central* in G if $\mathrm{ecc}(v) = \mathrm{rad}(G)$.

An *independent set* of a graph G, is a set $S \subseteq V(G)$, such that the induced subgraph H on the vertices of S is empty. The *independence number* of G, denoted $\alpha(G)$, is the size of a maximum independent set in G.

A *tree* is a connected acyclic graph. An acyclic graph that is disconnected is known as a *forest*.

A *comb* is a graph obtained by adding a leaf to every vertex of a path, shown in Figure 1.1. A *broken comb* is a connected subgraph of a comb that has a unique pair of leaves that realize the diameter, shown in Figure 1.2. A unique pair of leaves that realize the diameter of a broken comb are called *antipodal vertices*. A *bijacent vertex* is a degree two vertex with two degree two neighbors.

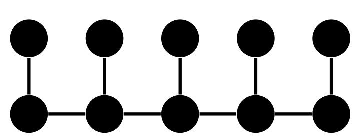

Figure 1.1 Comb formed from a P_5.

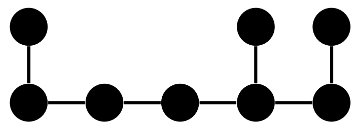

Figure 1.2 Broken Comb.

A *rooted tree* is a tree in which one vertex is designated the *root*. In a rooted tree, the *parent* of a vertex is the vertex adjacent to it on the path to the root, note that every vertex except the root has a unique parent. A *child* of a vertex v is any vertex for which v is its parent. A *complete d-ary tree* is a rooted tree in which each vertex, except the leaves, has d children. A complete 2-ary tree is also referred to as a *complete binary tree* and in this dissertation $\mathcal{B}_n$ will denote the binary tree on $2^{n+1}-1$ vertices. Figure 1.3 shows $\mathcal{B}_2$.

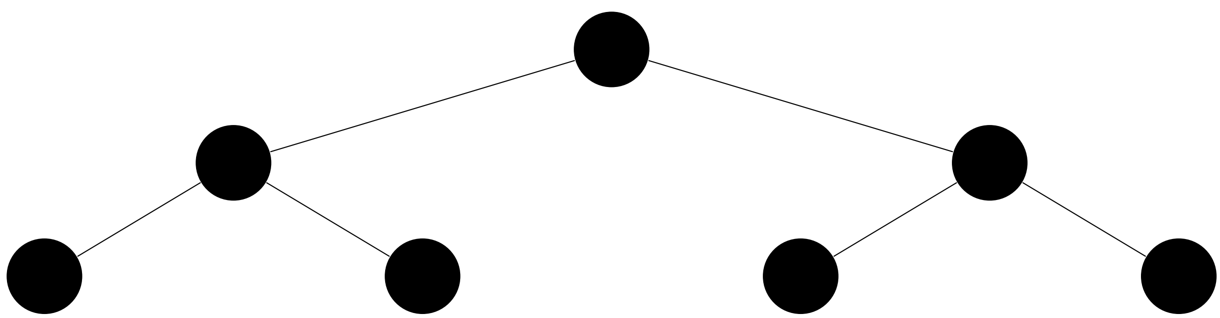

Figure 1.3 Binary tree $\mathcal{B}_2$.

If $G = (V, E)$ and $H = (V', E')$ then the *Cartesian product*, written $G \Box H$, has vertex set $\{(x, y') \mid x \in V \text{ and } y' \in V'\}$ and (x, y') and (u, v') are adjacent in $G \Box H$ if either $x = u$ and $y'v' \in E'$ or $y' = v'$ and $xu \in E$.

In this dissertation, P_n denotes the path graph on n vertices. The vertex set of $P_m \Box P_n$ is given by $\{v_{i,j} \mid 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$. This graph is visually represented as a grid and can be thought of as having m rows and n columns. Further, $v_{i,j}$ can be found at the intersection of the ith row and jth column of $P_m \Box P_n$. This convention allows for the computation of distances in grid graphs based on the subscripts of the vertices. In particular, if $v_{i,j}$ and $v_{\ell,k}$ are in $P_m \Box P_n$ then $\mathrm{d}(v_{i,j}, v_{\ell,k}) = |i - \ell| + |j - k|$.

The *hypercube* on 2 vertices is denoted Q_1 and $Q_1 = K_2$. For larger hypercubes, define $Q_n = Q_{n-1} \square K_2$ where Q_n denotes the n-dimensional hypercube. Note that Q_n has 2^n vertices. Label the vertices with binary strings of length n such that the distance between two vertices is equal to the number of bits they differ by. Observe that the diameter of Q_n is n. Figure 1.4 shows the hypercube Q_3.

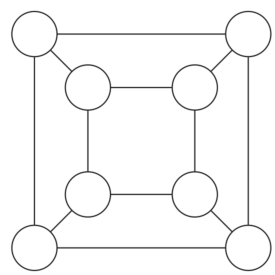

Figure 1.4 Hypercube Q_3.

A *blow-up* of a graph is obtained by replacing each vertex with a finite set of copies so that the copies of two vertices are adjacent if and only if the original vertices were adjacent. A *balanced blow-up* is a blow-up in which every finite set of copies is the same size. A balanced blow-up of uniform size ℓ will be referred to as a ℓ-blow-up. Figure 1.5 shows a path on 5 vertices and Figure 1.6 shows the 3-blow-up of that graph.

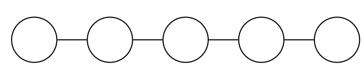

Figure 1.5 Path graph P_5

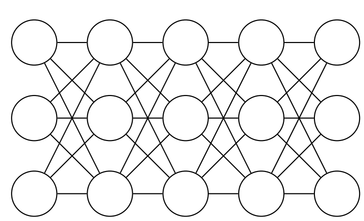

Figure 1.6 3-blow-up of P_5

A *k-term arithmetic progression of a graph G*, k-AP, is a set of k vertices of G of the form $\{v_1, v_2, \ldots, v_k\}$, where $\mathrm{d}(v_i, v_{i+1}) = d$ for all $1 \leq i < k$ and $d < \infty$. Throughout the remainder of this dissertation the order the set is written will denote the order of the arithmetic progression. A k-term arithmetic progression is *degenerate* if $v_i = v_j$ for any $i \neq j$. In Figure 1.7 the non-degenerate 3-APs are $\{x, u, v\}$, $\{x, u, w\}$, $\{u, v, w\}$, $\{v, u, w\}$, $\{u, w, v\}$, $\{v, x, w\}$, $\{v, u, x\}$, $\{w, u, x\}$, $\{w, v, u\}$, $\{w, u, v\}$, $\{v, w, u\}$, and $\{w, x, v\}$. However, if $\{x, u, v\}$ is a rainbow 3-AP, then so is $\{v, u, x\}$, so the reversal's of the progressions are also valid progressions, but only one direction needs to be considered.

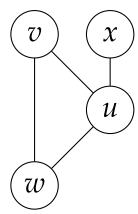

Figure 1.7 Paw graph

An *exact r-coloring of a graph G* is a surjective function $c : V(G) \to \{1, 2, \ldots, r\}$. A set of vertices $S \subseteq V(G)$ is *rainbow* under coloring c, if for any $v_i, v_j \in S$, $c(v_i) \neq c(v_j)$ when $v_i \neq v_j$. Note that degenerate k-APs will not be rainbow. Given a set of vertices $S \subseteq V(G)$, $c(S) = \{c(v) | v \in S\}$, is the set of colors used on the vertices of S. In Figure 1.8 the rainbow 3-APs are $\{x, u, w\}$ and $\{v, x, w\}$.

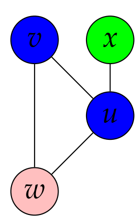

Figure 1.8 Paw graph under an exact 3-coloring.

The *anti-van der Waerden number of a graph G*, denoted by $\mathrm{aw}(G, k)$, is the least positive integer r such that every exact r-coloring of G contains a non-degenerate rainbow k-AP. If G has n vertices and no coloring of G contains non-degenerate k-APs, then $\mathrm{aw}(G,k) = n+1$. For a graph G, if $\mathrm{aw}(G,k) = r$, then an *extremal coloring* is an exact $(r-1)$-coloring of G that avoids rainbow 3-APs.

1.2.4 Ramsey Numbers

Though the majority of this dissertation focuses on colorings of the vertices to determine the anti-van der Waerden number of a graph, the edges of a graph may also be colored.

An *exact r-edge coloring* of a graph G is a surjective function from $c : E(G) \to [r]$ where $\{1, \ldots, r\}$ is a set of r colors, this is more typically called an *edge coloring* or an *r-edge coloring*.

Let G be an edge colored graph and H be a subgraph of G, H is *edge monochromatic* if every edge of H is the same color. The *Ramsey number*, $R(k,\ell)$, for positive integers k and ℓ, is the smallest n such that for every 2-edge coloring of K_n there exists a monochromatic subgraph isomorphic to K_k in color 1 or a monochromatic subgraph isomorphic to K_ℓ in color 2.

Let $k_1, k_2, \ldots, k_r$ be positive integers. The Ramsey number, $R(k_1, k_2, \ldots, k_r)$, is the smallest n such that every r-edge coloring of K_n contains an edge monochromatic subgraph isomorphic to K_{k_i} in color i for some $i = 1, \ldots, r$.

Let $G_1, G_2, \ldots, G_r$ be graphs. The Ramsey number, $R(G_1, G_2, \ldots, G_r)$ is the smallest n such that every r-edge coloring of K_n contains an edge monochromatic subgraph isomorphic to G_i in color i for some $i \in \{1, 2, \ldots, r\}$. If $G_i = G_j$ for all $1 \leq i, j \leq r$, this can be written $R_r(G)$.

1.2.5 Rainbow Numbers and anti-Schur Numbers

Both the concept of rainbow numbers and anti-Schur numbers are closely related to anti-van der Waerden numbers.

For a fixed integer k, a *triple* (x_1, x_2, x_3) is any three elements in G which are a solution to $x_1 + x_2 = kx_3$. When $k = 1$, these are referred to as *Schur triples*. A triple is called a *rainbow triple* under a coloring c when $c(x_1) \neq c(x_2)$, $c(x_1) \neq c(x_3)$, and $c(x_2) \neq c(x_3)$. A coloring is called *rainbow-free* when there does not exist a rainbow triple in G under c. For positive integer k, the *rainbow number* of a mathematical object G, denoted $\mathrm{rb}(G,k)$, is the least positive integer r such that every exact r-coloring of G contains a rainbow triple (x_1, x_2, x_3) such that $x_1 + x_2 = kx_3$ and $x_1, x_2, x_3 \in G$. If such an integer does not exist, $\text{rb}(G, k) = |G| + 1$. A *maximum* coloring is a rainbow-free r-coloring of G where $r = \text{rb}(G, k) - 1$.

For positive integers n and k, the *anti-Schur number*, denoted $\text{as}(G, k)$, is the least positive integer r such that every exact r-coloring of the mathematical object G contains a rainbow set $\{x_1, x_2, \ldots, x_k\}$ such that $\sum_{i=1}^{k-1} x_i = x_k$ and $x_j \in G$ for $1 \leq j \leq k$. If such an integer does not exist, $\text{as}(G, k) = |G| + 1$.

CHAPTER 2. THE ANTI-VAN DER WAERDEN NUMBER OF GROUPS

In this chapter, the anti-van der Waerden number of groups is discussed. The primary results are for both $[n]$ and $\mathbb{Z}_n$. A more thorough examination of the integers modulo n and the integers can be found in the author's Masters Thesis [26]. This chapter will summarize the results there and highlight a few key results and proofs. Then the anti-van der Waerden for abelian and non-abelian groups are discussed.

2.1 The anti-van der Waerden Number of $\mathbb{Z}_n$ and $[n]$

The specific problem of determining anti-van der Waerden numbers for $[n]$ and $\mathbb{Z}_n$ was studied by Butler et al. in [7]. It is proved in [7] that for $k \geq 4$, $\operatorname{aw}([n], k) = n^{1-o(1)}$ and $\operatorname{aw}(\mathbb{Z}_n, k) = n^{1-o(1)}$. These results are obtained using results of Behrend [3] and Gowers [12] on the size of a subset of $[n]$ with no k-AP. Butler et al. also expand upon the results of [16] by determining $\operatorname{aw}(\mathbb{Z}_n, 3)$ for all values of n. These results were generalized to all finite abelian groups in [31]. Butler et al. also provide bounds for $\operatorname{aw}([n], 3)$, as well as many exact values (see Table 4.1). In [4] the exact value of $\operatorname{aw}([n], 3)$ is determined and shows the existence of extremal colorings of $[n]$ that are unitary, (Theorem 5), which answers questions posed in [7].

In [7, Theorem 1.6] it is shown that $3 \leq \operatorname{aw}(\mathbb{Z}_p, 3) \leq 4$ for every prime number p and that if $\operatorname{aw}(\mathbb{Z}_p, 3) = 4$ then $p \geq 17$. Furthermore, it is shown that the value of $\operatorname{aw}(\mathbb{Z}_n, 3)$ is determined by the values of $\operatorname{aw}(\mathbb{Z}_p, 3)$ for the prime factors p of n, Theorem 2 shows this result with notational changes and is used to prove Lemma 3.

Theorem 2. *[7] Let n be a positive integer with prime decomposition $n = 2^{e_0} p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s}$ for $e_i \geq 0$, $i = 0, \ldots, s$, where primes are ordered so that $\operatorname{aw}(\mathbb{Z}_{p_i}, 3) = 3$ for $1 \leq i \leq \ell$ and $\operatorname{aw}(\mathbb{Z}_{p_i}, 3) = 4$ for*

$\ell+1 \leq i \leq s$. Then

$$\operatorname{aw}(\mathbb{Z}_n,3)=\begin{cases} 2+\sum_{j=1}^{\ell} e_j+\sum_{j=\ell+1}^{s} 2e_j, & \textit{if } n \textit{ is odd}\\ 3+\sum_{j=1}^{\ell} e_j+\sum_{j=\ell+1}^{s} 2e_j, & \textit{if } n \textit{ is even.}\end{cases}$$

Lemmas 3 and 4 are used in the proof of Theorem 5.

Lemma 3. *[4] If $n \geq 3$, then $\operatorname{aw}(\mathbb{Z}_n,3) \leq \lceil \log_3 n \rceil + 2$ with equality if and only if $n = 3^j$ or $2 \cdot 3^j$ for $j \geq 1$.*

Lemma 4. *[4] If N is an integer and c is an exact r-coloring of $[N]$ with no rainbow 3-AP, where 1 and N are colored uniquely, then either the coloring c is special or $|\{c(x) : x \equiv i \pmod 3 \text{ and } x \in [N]\}| \geq r-1$ for $i = 1$ or $i = N$.*

Given positive integers n and m, define the function f as follows:

$$f(n)=\begin{cases} m+2, & \text{if } n=3^m\\ m+3, & \text{if } n\neq 3^m \text{ and } 7\cdot 3^{m-2}+1\leq n\leq 21\cdot 3^{m-2}.\end{cases}$$

Theorem 5 is proved by showing that $\operatorname{aw}([n],3) = f(n)$ for all n. Throughout the proof $\pmod 3$ is mostly dropped, although all equivalences will happen modulo 3.

Theorem 5. *[4] For all integers $n \geq 2$,*

$$\operatorname{aw}_u([n],3)=\operatorname{aw}([n],3)=\begin{cases} m+2, & \textit{if } n=3^m\\ m+3, & \textit{if } n\neq 3^m \textit{ and } 7\cdot 3^{m-2}+1\leq n\leq 21\cdot 3^{m-2}.\end{cases}$$

Proof. First, it is shown that $f(n) \leq \operatorname{aw}_u([n],3)$ by inductively constructing a unitary coloring of $[n]$ with $f(n)-1$ colors and no rainbow 3-AP. The result is true for $n = 1,2,3$, by definition. Suppose $n > 3$ and that the result holds for all positive integers less than n. Let $n = 3h - s$, where $s \in \{0,1,2\}$ and $2 \leq h < n$.

Let $r = \operatorname{aw}_u([h],3)$. So there is an exact unitary $(r-1)$-coloring c of $[h]$ with no rainbow 3-AP. Let *red* be a color not used in c. Define the coloring c_1 of $[n]$ such that if $x \equiv 1 \pmod 3$, then $c_1(x) = c(\frac{x+2}{3})$, otherwise color x *red*. When $s \neq 0$, define the coloring c_2 of $[n]$ as follows:

- if $x \not\equiv 0 \pmod 3$, then $c_2(x) = red$,
- if $x \equiv 0 \pmod 3$, then
  - $c_2(x) = c(\frac{x}{3} + 1)$, if $c(h)$ is the only unique color in c,
  - $c_2(x) = c(\frac{x}{3})$, otherwise.

Notice that c_2 is a unitary $\text{aw}_u([h-1], 3)$-coloring when $s \neq 0$ and c_1 is a unitary r-coloring of $[n]$. Now consider a 3-AP $\{a, b, 2b-a\}$ in $[n]$. If $a \equiv b \not\equiv 1$, then a and b are colored with *red*, and so the 3-AP is not a rainbow. If $a \equiv b \equiv 1$, then $2b - a \equiv 1$, so this set corresponds to a 3-AP in $[h]$ with coloring c, and hence the 3-AP is not rainbow. If $a \not\equiv b$, then $2b - a$ is not congruent to a or b, so two of the terms of the 3-AP are colored with *red*, and hence the 3-AP is not rainbow under c_1. Similarly, this 3-AP is not rainbow under c_2. Therefore, c_1 and c_2 are unitary colorings of $[n]$ with no rainbow 3-AP. Also note that $\text{aw}_u([n], 3) \geq \text{aw}_u([h], 3) + 1$ under c_1 and $\text{aw}_u([n], 3) \geq \text{aw}_u([h-1], 3) + 1$ under c_2. Now proceed with two cases determined by $\frac{n}{3}$.

Case 1. First suppose $7 \cdot 3^{m-2} + 1 \leq n \leq 3^m - 3$ or $3^m \leq n \leq 21 \cdot 3^{m-2}$.

By the induction hypothesis and using the coloring c_1,

$$\text{aw}_u([n], 3) \geq \text{aw}_u([h], 3) + 1 \geq f(h) + 1 = f(n).$$

Case 2. Suppose $n = 3^m - t$ where $t \in \{1, 2\}$.

Notice that $h = 3^{m-1}$, so by induction and using coloring c_2,

$$\text{aw}_u([n], 3) \geq \text{aw}_u([h-1], 3) + 1 \geq f(h-1) + 1 = f(3^{m-1} - 1) + 1 = (m+2) + 1 = f(n).$$

The upper bound, $\text{aw}([n], 3) \leq f(n)$, is also proved by induction on n. For small n, the result follows from Table 4.1. Assume the statement is true for all values less than n, and let $7 \cdot 3^{m-2} + 1 \leq n \leq 21 \cdot 3^{m-2}$ for some m. Let $\text{aw}([n], 3) = r + 1$, so there is an exact r-coloring $\hat{c}$ of $[n]$ with no rainbow 3-AP. It remains to show that $r \leq f(n) - 1$. Let $[n_1, n_2, \ldots, n_N]$ be the shortest interval in $[n]$ containing all r colors under $\hat{c}$. Define c to be an r-coloring of $[N]$ so that $c(j) = \hat{c}(n_j)$ for

$j \in \{1, \ldots, N\}$. By minimality of N the colors of 1 and N are unique. If $[N]$ has at least $r-1$ colors congruent to 1 or N, then $[n]$ has at least $r-1$ colors congruent to n_1 or n_N, respectively, so $r \leq \mathrm{aw}(\lfloor \frac{n}{3} \rfloor, 3)$ and by induction $r \leq f(\lfloor \frac{n}{3} \rfloor) \leq f(n) - 1$. So suppose that is not the case, then by Lemma 4 we have that the coloring c is special.

Let $N = 7q + 1$ for some $q \geq 1$, and let the 8-AP in this special coloring be $\{1, r_1, r_2, b_1, r_3, b_2, b_3, N\}$, where r_1, r_2, r_3 are the only integers colored *red*, b_1, b_2, b_3 are the only integers colored *blue* and $q = r_1 - 1$. If $n \geq 9q$, then the 8-AP can be extended to a 9-AP in n by adding the 9th element to either the beginning or the ending. Without loss of generality, suppose $\{1, r_1, r_2, b_1, r_3, b_2, b_3, N, 2N - b_3\}$ correspond to a 9-AP in $[n]$. Since the coloring has no rainbow 3-AP, the color of $2N - b_3$ is *blue* or $c(N)$, so there is a 4-coloring of this 9-AP. However, $\mathrm{aw}([9], 3) = 4$ and hence there is a rainbow 3-AP in this 9-AP which is in turn a rainbow 3-AP in $[n]$. Therefore, $n \leq 9q - 1$.

By uniqueness of the *red* colored integer r_1 in interval $\{1, \ldots, r_2 - 1\}$, the colors of integers in interval $\{r_1 + 1, \ldots, r_2 - 1\}$ is the same as the reversed colors of integers in $\{2, \ldots, r_1 - 1\}$, i.e. $c(r_1 + i) = c(r_1 - i)$ for $i = 1, \ldots, q - 1$. Similarly, coloring of integers in interval $\{r_2 + 1, \ldots, b_1 - 1\}$ is the reversed of the coloring of integers in interval $\{r_1 + 1, \ldots, r_2 - 1\}$, and so on. This gives a rainbow 3-AP-free $(r-2)$-coloring of $\mathbb{Z}_{2q}$. Therefore, $r - 2 \leq \mathrm{aw}(\mathbb{Z}_{2q}, 3) - 1$.

If $q = 3^i$ for some i, then n can not be a power of 3 because $7 \cdot 3^i + 1 \leq n \leq 9 \cdot 3^i - 1$. Suppose $n = 3^m$, then $2q$ is not twice a power of 3 and clearly $2q$ is not a power of 3. Therefore, by Lemma 3 we have

$$
\begin{aligned}
r &\leq \mathrm{aw}(\mathbb{Z}_{2q}, 3) + 1 \leq \lceil \log_3(2q) \rceil + 2 \leq \lceil \log_3(\frac{2n}{7}) \rceil + 2 \\
&= \lceil \log_3(\frac{2 \cdot 3^m}{7}) \rceil + 2 = m + 1 \leq f(n) - 1.
\end{aligned}
$$

Suppose now that $n \neq 3^m$. If $q = 3^i$ for some i then $i \leq m - 2$. Otherwise, if $i \geq m - 1$ then $q \geq 3^{m-1} \geq \frac{n}{7}$ which contradicts the fact that $q < \frac{n}{7}$. Therefore, $2q \leq 2 \cdot 3^{m-2} = 18 \cdot 3^{m-4}$ and so by induction and Lemma 3,

$$
r \leq \mathrm{aw}(\mathbb{Z}_{2q}, 3) + 1 = \mathrm{aw}([2q], 3) + 1 \leq m + 2 \leq f(n) - 1.
$$

If q is not a power of 3, then again using Lemma 3, $r \leq \mathrm{aw}(\mathbb{Z}_{2q}, 3) + 1 \leq \mathrm{aw}([2q], 3)$. Notice that $6 \cdot 3^{m-3} + \frac{2}{7} \leq \frac{2}{7}n \leq 18 \cdot 3^{m-3}$, and so $\mathrm{aw}([2q], 3) \leq m + 2$ by induction. Therefore, $r \leq m + 2 \leq f(n) - 1$. □

2.2 Groups

The anti-van der Waerden number has been studied on abelian groups besides $\mathbb{Z}_n$ in [31]. The preliminary study of the anti-van der Waerden number of non-abelian groups has been examined in [8]. These results are detailed in the authors masters thesis, [26]. However, some of the results are highlighted here.

2.2.1 Abelian Groups

Theorem 6. [31] *For any positive odd integer n and finite abelian group G,*

$$\mathrm{aw}_u(G \times \mathbb{Z}_n, 3) = \mathrm{aw}_u(G, 3) + \mathrm{aw}_u(\mathbb{Z}_n, 3) - 2.$$

Corollary 7. [31] *Let n be the largest odd divisor of the order of a group G, then there exists a finite abelian group G' such that the order of G' is a power of 2 and*

$$\mathrm{aw}_u(G, 3) = \mathrm{aw}_u(G', 3) + \mathrm{aw}_u(\mathbb{Z}_n, 3) - 2.$$

Theorem 8. [31] *Let $1 \leq i \leq s$ and let m_i be a positive integer, then*

$$\mathrm{aw}_u(\mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \cdots \times \mathbb{Z}_{2^{m_s}}, 3) = s + 2.$$

2.2.2 Non-Abelian Groups

Let G be a non-abelian group with operation $*$. A *right k-term arithmetic progression* of G is a set of the form $\{a, a * d, \ldots, a * d^{k-1}\}$, where $a, d \in G$. A *left k-term arithmetic progression* of G is a set of the form $\{a, d * a, \ldots, d^{k-1} * a\}$, where $a, d \in G$. Proposition 9 allows us to only consider right arithmetic progressions when determining $\mathrm{aw}(G, k)$ when G is non-abelian.

Proposition 9. *[8] The set of right arithmetic progressions is equal to the set of left arithmetic progressions.*

Table 2.1 Quaternion Multiplication Table

$\cdot$	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

For $n \geq 3$, the dihedral group $\mathcal{D}_n$, is the set of symmetries of a regular n-gon with the operation of composition. $\mathcal{D}_n$ is composed of n rotations and n reflections. We denote $\mathcal{R}_n$ as the set of rotations of $\mathcal{D}_n$ and $\mathcal{F}_n$ as the set of reflections of $\mathcal{D}_n$.

The subgroup $\mathcal{R}_n$ is isomorphic to $\mathbb{Z}_n$, so let ϕ be an isomorphism from $\mathcal{R}_n$ to $\mathbb{Z}_n$. Let $f \in \mathcal{F}_n$ and the function $\phi_f : \mathcal{F}_n \to \mathbb{Z}_n$ be defined by $\phi_f(x) = \phi(fx)$. The function ϕ_f is an isomorphism between the set of k-term arithmetic progressions in $\mathcal{F}_n$ and the set of k-term arithmetic progressions in $\mathbb{Z}_n$.

Theorem 10. *[8] Let $n \in \mathbb{N}$, then* $\operatorname{aw}(\mathcal{D}_n, k) = 2\operatorname{aw}(\mathbb{Z}_n, k) - 1$.

The quaternions, denoted Q_8, is the set $\{1, -1, i, -i, j, -j, k, -k\}$ with quaternion multiplication as the operation as seen in Table 2.1. In [8] they determined $\operatorname{aw}(Q_8, 3)$ by finding a lower bound and an upper bound that are equal resulting with $\operatorname{aw}(Q_8, 3) = 5$ shown in Theorem 11.

Theorem 11. *[8]* $\operatorname{aw}(Q_8, 3) = 5$.

CHAPTER 3. THE ANTI-VAN DER WAERDEN NUMBER OF GRAPHS

In this chapter, the anti-van der Waerden number is investigated for graphs. There is a direct connection between studying the anti-van der Waerden number of $[n]$ and $\mathbb{Z}_n$ and studying the anti-van der Waerden number of graphs.

In Section 3.1, the motivation for studying the anti-van der Waerden number of graphs is given as well as some preliminary results. In Section 3.2, bounds are given for graphs using distance parameters such as radius and diameter. In Section 3.3, the anti-van der Waerden number is bounded for trees. In Section 3.4, the anti-van der Waerden number of Cartesian products of graphs is bounded. The anti-van der Waerden number for some graph families is determined in Section 3.5.

3.1 Motivation and Observations

The motivation for examining the anti-van der Waerden number of graphs originates from extending results on the anti-van der Waerden number of $[n]$ and $\mathbb{Z}_n$ to paths and cycles, respectively. Notice that the set of arithmetic progressions on $[n]$ is isomorphic to the set of non-degenerate arithmetic progressions on P_n and the set of arithmetic progressions on $\mathbb{Z}_n$ is isomorphic to the set of non-degenerate arithmetic progressions on C_n. Therefore, considering the anti-van der Waerden number of $[n]$ or $\mathbb{Z}_n$ is equivalent to studying the anti-van der Waerden number of paths or cycles respectively.

In [7] Butler et al. determineed $\mathrm{aw}(C_n, 3)$ as seen in Theorem 13. This theorem was generalized in [31]. In [7], they also obtained bounds on $[n]$ and conjectured the result that was proved in [4]. This result on $[n]$ is adapted to paths in Theorem 12.

Theorem 5 is generalized to a graph theoretic result in Theorem 12 and Theorem 2 is generalized in a similar way in Theorem 13.

Theorem 12. *[4] If $n \geq 3$ and $7 \cdot 3^{m-2} + 1 \leq n \leq 21 \cdot 3^{m-2}$, then*

$$\operatorname{aw}([n],3) = \operatorname{aw}(P_n,3) = \begin{cases} m+2 & \text{if } n = 3^m \\ m+3 & \text{otherwise.} \end{cases}$$

In [7, Theorem 1.6] it is shown that $3 \leq \operatorname{aw}(\mathbb{Z}_p, 3) \leq 4$ for every prime number p and that if $\operatorname{aw}(\mathbb{Z}_p, 3) = 4$, then $p \geq 17$. Furthermore, it is shown that the value of $\operatorname{aw}(\mathbb{Z}_n, 3)$ is determined by the values of $\operatorname{aw}(\mathbb{Z}_p, 3)$ for the prime factors p of n. The notation has been changed, but this result is given in Theorem 13.

Theorem 13. *[7] Let n be a positive integer with prime decomposition $n = 2^{e_0} p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s}$ for $e_i \geq 0$, $i = 0, \ldots, s$, where primes are ordered so that $\operatorname{aw}(\mathbb{Z}_{p_i}, 3) = 3$ for $1 \leq i \leq \ell$ and $\operatorname{aw}(\mathbb{Z}_{p_i}, 3) = 4$ for $\ell + 1 \leq i \leq s$. Then,*

$$\operatorname{aw}(\mathbb{Z}_n,3) = \operatorname{aw}(C_n,3) = \begin{cases} 2 + \sum_{j=1}^{\ell} e_j + \sum_{j=\ell+1}^{s} 2e_j, & \text{if } n \text{ is odd} \\ 3 + \sum_{j=1}^{\ell} e_j + \sum_{j=\ell+1}^{s} 2e_j, & \text{if } n \text{ is even.} \end{cases}$$

The anti-van der Waerden number is not a monotone parameter. The inequality

$$\operatorname{aw}(C_n,3) = \operatorname{aw}(\mathbb{Z}_n,3) \leq \operatorname{aw}([n],3) = \operatorname{aw}(P_n,3),$$

yields examples of this parameter not being monotone. For a simple example, consider C_4 and P_4.

The lower bound for Observation 14 is easy to see, and the upper bound is due to the fact that it is possible to color each vertex of a graph uniquely and fail to have a rainbow k-AP. A *complete graph* on n vertices, written K_n, is a graph such that there is an edge between every pair of vertices. Complete graphs realize the lower bound of Observation 14, namely $\operatorname{aw}(K_n, k) = k$. An *empty graph* on n vertices is a graph on n vertices with no edges. Empty graphs realize the upper bound of Observation 14.

Observation 14. *[27] Let G be a graph on n vertices, then $k \leq \operatorname{aw}(G, k) \leq n + 1$.*

Notice that by using Observation 14 every coloring will need to use at least k colors to contain a rainbow k-term arithmetic progression. With this in mind, every coloring used in the proofs of the results will start with at least k colors.

A subgraph H of G is *isometric* if for all $u, v \in V(H)$ $\mathrm{d}_H(u,v) = \mathrm{d}_G(u,v)$.

Lemma 15. *[22] If H is an isometric subgraph of G, then a k-AP in H is a k-AP in G. If there exists a k-AP in G that only contains vertices of H, then it is also a k-AP in H.*

Proof. Let $\{v_1, v_2, \ldots, v_k\}$ be a k-AP in H. Since this is a k-AP, then $\mathrm{d}_H(v_i, v_{i+1}) = d$ for $1 \le i \le k-1$. By the definition of isometric subgraph, $\mathrm{d}_H(v_i, v_{i+1}) = \mathrm{d}_G(v_i, v_{i+1})$. Hence, $\{v_1, v_2, \ldots, v_k\}$ is a k-AP in G.

Now suppose $\{v_1, v_2, \ldots, v_k\}$ is a k-AP in G and $v_i \in V(H)$ for $1 \le i \le k$. Since $\{v_1, v_2, \ldots, v_k\}$ is a k-AP in G, then $\mathrm{d}_G(v_i, v_{i+1}) = d$ for $1 \le i \le k-1$. Since H is an isometric subgraph, $\mathrm{d}_G(v_i, v_{i+1}) = \mathrm{d}_H(v_i, v_{i+1})$ for all $1 \le i \le k-1$, and therefore, $\{v_1, v_2, \ldots, v_k\}$ is a k-AP in H. □

Proposition 16. *[22] If H is an isometric subgraph of G and c is an exact r-coloring of G that avoids rainbow k-APs, then H contains at most $\mathrm{aw}(H,k) - 1$ colors.*

Proof. Suppose by way of contradiction, $|c(H)| \ge \mathrm{aw}(H,k)$. This implies H has a rainbow k-AP, namely $\{v_1, v_2, \ldots, v_k\}$, since every $\mathrm{aw}(H,k)$-coloring of H must have a rainbow k-AP by definition. By Lemma 15, $\{v_1, v_2, \ldots, v_k\}$ is also a k-AP in G, a contradiction. Hence, any isometric subgraph H of G has at most $\mathrm{aw}(H,k) - 1$ colors. □

Note that Proposition 16 ensures that whenever there exists a rainbow 3-AP in an isometric subgraph of G, there is a corresponding rainbow 3-AP in G. Therefore, when dealing with isometric subgraphs, the anti-van der Waerden number is a monotone parameter.

A graph G is *connected* if for each pair of vertices $u, v \in V(G)$ there exists a path from u to v in G. A graph that is not connected is called *disconnected*. A *connected component* of G is a maximal connected subgraph of G.

Proposition 17. *[27] If G is disconnected with connected components $\{G_i\}_{i=1}^{\ell}$, then*

$$\mathrm{aw}(G,k) = 1 + \sum_{i=1}^{\ell} (\mathrm{aw}(G_i,k) - 1).$$

Proof. By definition each component G_i can be colored with $\mathrm{aw}(G_i, k) - 1$ colors and avoid rainbow 3-APs. To maximize the number of colors, choose the colors so that $c(G_i) \cap c(G_j) = \emptyset$ for $i \neq j$. Thus, rainbow 3-APs have been avoided in G using $\sum_{i=1}^{\ell} (\mathrm{aw}(G_i, k) - 1)$ colors, so

$$1 + \sum_{i=1}^{\ell} (\mathrm{aw}(G_i, k) - 1) \leq \mathrm{aw}(G, k).$$

However, if $1 + \sum_{i=1}^{\ell} (\mathrm{aw}(G_i, k) - 1)$ colors are used on G, then some component G_i must use at least $\mathrm{aw}(G_i, k)$ colors. This means

$$\mathrm{aw}(G, k) \leq 1 + \sum_{i=1}^{\ell} (\mathrm{aw}(G_i, k) - 1).$$

□

Proposition 17 shows that studying disconnected graphs is equivalent to studying each of its connected components. Therefore, for the remainder of this dissertation only connected graphs will be considered.

3.2 Distance Parameters

In this section, anti-van der Waerden numbers are determined and bounded in terms of distance parameters. A vertex v in graph G is *dominating* if v is adjacent to each other vertex of G. The *eccentricity* of vertex v in graph G is $\mathrm{ecc}(v) = \max\{\mathrm{d}(v, u) \mid u \in V(G)\}$. The *radius* of graph G is $\mathrm{rad}(G) = \min\{\mathrm{ecc}(v) \mid v \in V(G)\}$ and the *diameter* is $\mathrm{diam}(G) = \max\{\mathrm{ecc}(v) \mid v \in V(G)\}$. A vertex v is *central* in G if $\mathrm{ecc}(v) = \mathrm{rad}(G)$.

Observation 18. *If graph G has a dominating vertex, then* $\mathrm{aw}(G, 3) = 3$.

Notice if a graph G has a dominating vertex, then as soon as there are 3 colors in G, it will have a rainbow 3-AP where the dominating vertex is the center of the progression. This yields the result in Observation 18.

3.2.1 Radius Parameters

In this subsection, bounds for the anti-van der Waerden number using the radius are determined.

Proposition 19. *[27] For all connected graphs G,*

$$\mathrm{aw}(G,3) \leq \mathrm{rad}(G)+2.$$

Proof. Let v be a central vertex of G. For $1 \leq i \leq \mathrm{rad}(G)$, let L_i be the set of vertices that are distance i from v. In any exact $(\mathrm{rad}(G)+2)$-coloring of G, there exists an L_i that has two colors that are different from the color of v. If u and w are two such vertices in L_i, then $\{u,v,w\}$ is a rainbow 3-AP. □

Proposition 19 and Observation 14 give the fact that if $\mathrm{rad}(G)=1$, then $\mathrm{aw}(G,3)=3$. However, Proposition 19 can be improved when $\mathrm{rad}(G)\geq 3$, as seen in Proposition 20.

Proposition 20. *[27] If G is a connected graph with* $\mathrm{rad}(G)\geq 3$, *then*

$$\mathrm{aw}(G,3) \leq \mathrm{rad}(G)+1.$$

Proof. Let G be a connected graph with radius $\mathrm{rad}(G)\geq 3$ and v be a central vertex of G. Let c be an exact $(\mathrm{rad}(G)+1)$-coloring of G that avoids rainbow 3-APs. Without loss of generality, let $c(v)=$ *red* and L_i be the set of vertices that are distance i from v (see Figure 3.1). Note that if $v_1,v_2 \in L_i$ where $c(v_1),c(v_2)\neq$ *red* and $c(v_1)\neq c(v_2)$, then $\{v_1,v,v_2\}$ is a rainbow 3-AP. Since there are $\mathrm{rad}(G)$ sets L_i, each non-*red* color appears in exactly one L_i. Let $b\in L_1$, $g\in L_2$ and $p\in L_3$ be vertices that are not colored *red*. There are no edges between b, g and p, otherwise there would exist a path on three vertices with three different colors which is a rainbow 3-AP. Therefore there exists a vertex colored *red*, r_1, in L_1 that is adjacent to g. If r_2 is adjacent to b or p, then there would exist a path on three vertices with three different colors. This implies there exists $r_2\in L_2$ that is colored *red* and is adjacent to p. If r_1 is adjacent to r_2, then $\{b,r_1,p\}$ is a rainbow 3-AP. Therefore there exists a vertex colored *red*, r_1', in L_1 that is adjacent to r_2. Then either $\{b,r_1',p\}$ or $\{p,b,g\}$ is a rainbow 3-AP depending on whether or not br_1' is an edge. □

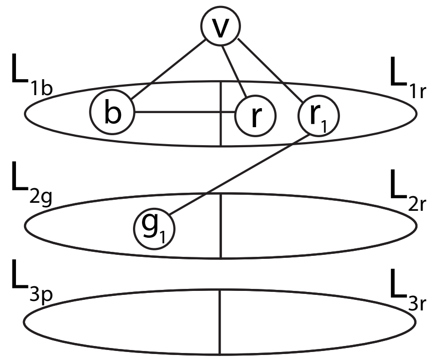

Figure 3.1 Levels of graph from central vertex v, edges drawn must exist.

3.2.2 Diameter Parameters

In this subsection, the anti-van der Waerden number of graphs are found and bounded using the graphs diameter.

Observation 21. *If we have unique diameter vertices (i.e. only one pair of vertices realize the diameter of the graph) and the diameter is odd, then* $4 \leq \operatorname{aw}(G,3)$ *by coloring the unique vertices with two different colors and every other vertex a third color.*

Though Proposition 22 is not difficult to prove, the impact is large.

Proposition 22. *[27] Let G be a connected graph such that* $\operatorname{diam}(G) = 2$*, then* $\operatorname{aw}(G,3) = 3$.

Proof. Since $\operatorname{diam}(G) = 2$, then $\operatorname{rad}(G)$ is either 1 or 2. By Observation 14, $3 \leq \operatorname{aw}(G,3)$. If $\operatorname{rad}(G) = 1$, then $\operatorname{aw}(G,3) = 3$ since there exists vertices u, v, w such that v is a central vertex and $c(v) \neq c(u)$, $c(v) \neq c(w)$ and $c(u) \neq c(w)$, so $\{u, v, w\}$ is a rainbow 3-AP. Assume $\operatorname{rad}(G) = \operatorname{diam}(G) = 2$ and c is an exact 3-coloring of G. Let u and v be adjacent vertices such that $c(u) \neq c(v)$. Let w be a vertex such that $c(w) \neq c(u)$ and $c(w) \neq c(v)$. If w is adjacent to u or v, there is a rainbow 3-AP. If w is not adjacent to u or v, then $\{u, w, v\}$ is a rainbow 3-AP. $\square$

Theorem 23 is a well known fact in graph theory.

Theorem 23. *Almost all graphs have* $\operatorname{diam}(G) = 2$.

Combining Proposition 22 and Theorem 23 leads to Corollary 24 which determines the anti-van der Waerden number for many graphs when $k = 3$.

Corollary 24. *Almost all graphs have* $\mathrm{aw}(G, 3) = 3$.

The *join* of G and H, denoted $G + H$, is the graph obtained by taking $G \cup H$ and adding all possible edges between $V(G)$ and $V(H)$. Note since all possible edges exist between $V(G)$ and $V(H)$ in $G + H$, then $\mathrm{diam}(G + H) \leq 2$.

Corollary 25. *[27] For graph G and H,*

$$\mathrm{aw}(G + H, 3) = 3.$$

3.3 Trees

In this section, anti-van der Waerden numbers are bounded for trees. Corollary 26 follows from Proposition 22 and with Proposition 27 trees with diameter less than 4 need not be considered for the remainder of this section.

Corollary 26. *If T is a tree with diameter* 2 *then* $\mathrm{aw}(T, 3) = 3$

Proposition 27. *If T is a tree with diameter* 3*, then* $\mathrm{aw}(T, 3) = 4$.

Proof. Let T be a tree such that $\mathrm{diam}(T) = 3$ and let u and v be vertices that are leaves of a longest path in T. Let c be an exact 3-coloring of T where

$$c(w) := \begin{cases} \textit{red} & \text{if } \mathrm{d}(u, w) = 3 \\ \textit{green} & \text{if } \mathrm{d}(v, w) = 3 \\ \textit{blue} & \text{otherwise.} \end{cases} .$$

Any 3-AP that starts with a leaf either repeats its leaf color or has two *blue* vertices, thus is not a rainbow 3-AP. All other 3-APs start at a *blue* vertex and clearly do not form a rainbow 3-AP, therefore $4 \leq \mathrm{aw}(T, 3)$. Notice that $\mathrm{rad}(T) = 2$, so by Proposition 19 $\mathrm{aw}(T, 3) \leq 4$, hence $\mathrm{aw}(T, 3) = 4$. $\square$

A path with all of the colors used on it will be useful in proofs later this section. The existence of such a path is shown in Lemma 28.

Lemma 28. *[27] In any coloring of a tree with no rainbow 3-APs, there exists a path that contains all of the colors.*

Proof. Suppose c is an exact r-coloring of a tree T with no rainbow 3-APs. Let T' be the smallest subtree of T that contains all r colors. If v is a leaf in T', then no other vertex in T' has color $c(v)$; otherwise, T' is not the smallest subtree containing all r-colors. If T' is not a path, then T' has at least three leaves, namely u, v and w. Without loss of generality $\mathrm{d}(u,v) \leq \mathrm{d}(v,w)$. The 3-AP $\{u,v,x\}$ is rainbow where x is the vertex on a shortest vw-path such that $\mathrm{d}(u,v) = \mathrm{d}(v,x)$. Since T contains no rainbow 3-AP, T' must be a path. □

The first time Lemma 28 is used is in the proof of Proposition 29.

Proposition 29. *[27] If T is a tree with $\mathrm{diam}(T) = d$, then*

$$\mathrm{aw}(T,3) \leq \begin{cases} \mathrm{aw}([d+1],3) & \text{if } d \neq 3^m \\ \mathrm{aw}([d+1],3)+1 & \text{if } d = 3^m \end{cases}.$$

Proof. Let T be a tree with $\mathrm{diam}(T) = d$ and c be an exact $(\mathrm{aw}(T,3)-1)$-coloring of T with no rainbow 3-APs. By Lemma 28, there exists a path P that contains every color. Therefore $\mathrm{aw}(T,3)-1 \leq \mathrm{aw}([|P|],3)-1$. This implies that $\mathrm{aw}(T,3) \leq \mathrm{aw}([|P|],3)$. Since $|P| \leq d+1$, then $\mathrm{aw}([|P|],3) \leq \mathrm{aw}([d+1],3)+1$, by Theorem 12. Furthermore, this can be improved to $\mathrm{aw}([|P|],3) \leq \mathrm{aw}([d+1],3)$ when $d \neq 3^m$. □

The remainder of this section focuses on the degree two vertices of trees and defines bijacent vertices which can bound the anti-van der Waerden number of a tree.

Propositions 30 and 31 have similar proofs and show that a tree with one or fewer degree two vertices has an anti-van der Waerden number of at most 4.

Proposition 30. *If T is a tree with $\mathrm{diam}(T) \geq 3$ and no vertex has degree two, then $\mathrm{aw}(T,3) \leq 4$.*

Proof. Let T be a tree such that $\operatorname{diam}(T) \geq 3$ and no vertex has degree two. Assume, for the sake of contradiction, that c is an exact 4-coloring of T that avoids rainbow 3-APs. Let P be a shortest path that contains all of the colors of c; such a path is guaranteed by Lemma 28. Let u and v be the ends of the path and note that the minimality of P means colors $c(u)$ and $c(v)$ do not show up anywhere else on the path. Without loss of generality, assume $c(u) = purple$ and $c(v) = red$ and let the other colors of c be *green* and *blue*. Also note that $\mathrm{d}(u, v)$ is odd, otherwise there is a rainbow 3-AP immediately. If the length of P is 3, then clearly there is a rainbow 3-AP, so assume the length of P is 4 or more. This means, without loss of generality, there are three consecutive vertices on P, say v_1, v_2, v_3, such that $v_1, v_3 \in N(v_2)$ with $c(v_1) = blue$, $c(v_2) = blue$ and $c(v_3) = green$ where $\mathrm{d}(u, v_1) < \mathrm{d}(u, v_2)$. Note v_2 has a neighbor, x that is not on P. Now $c(x)$ cannot be *purple* or *red* else $\{x, v_2, v_3\}$ is rainbow. Also, $c(x)$ cannot be *blue* or *green* else $\{x, u, v_3\}$ is rainbow or $\{x, v, v_1\}$ is rainbow respectively. In all cases there is a rainbow 3-AP and the result is verified. □

Proposition 31. *[27] If T is a tree with exactly one degree two vertex, then* $\operatorname{aw}(T, 3) \leq 4$.

Proof. Let T be a tree with exactly one degree two vertex. Assume, for the sake of contradiction, that c is an exact 4-coloring of T that avoids rainbow 3-APs. Let P be a shortest path that contains all of the colors of c; such a path is guaranteed by Lemma 28. Let u and v be the ends of the path and note that the minimality of P means colors $c(v)$ and $c(w)$ do not show up anywhere else on the path. Without loss of generality, assume $c(u) = purple$ and $c(v) = red$ and let the other colors of c be *green* and *blue*. Also note that $\mathrm{d}(u, v)$ is odd, otherwise there is a rainbow 3-AP immediately. If the length of P is 3, then clearly there is a rainbow 3-AP, so assume the length of P is 4 or more. This means, without loss of generality, there are three consecutive vertices on P, say v_1, v_2, v_3, such that $v_1, v_3 \in N(v_2)$ with $c(v_1) = blue$, $c(v_2) = blue$ and $c(v_3) = green$ where $\mathrm{d}(u, v_1) < \mathrm{d}(u, v_2)$. If v_2 is not the degree two vertex, then it has some neighbor x that is not on P. Now $c(x)$ cannot be *purple* or *red* else $\{x, v_2, v_3\}$ is rainbow. Also, $c(x)$ cannot be *blue* or *green* else $\{x, u, v_3\}$ is rainbow or $\{x, v, v_1\}$ is rainbow respectively. This means v_2 must be the degree two vertex. Now, if another consecutive sequence of vertices on P show up with corresponding color sequence *blue*, *blue*, *green* or *green*, *blue*, *blue*, then refer to the earlier argument to show a rainbow

3-AP exists. This means the colors of the vertices alternate between *blue* and *green* from v_1 to u and from v_3 to v (in which case the end of the path has a rainbow 3-AP) or the path has color *blue* from u to v_1 and has color *green* from v_3 to v. If $\mathrm{d}(u, v_2) \leq \mathrm{d}(v_3, v)$, then u and v_2 start a rainbow 3-AP that ends with a vertex of color *green*. On the other hand, v and v_3 start a rainbow 3-AP that ends with a vertex of color *blue*. In all cases there is a rainbow 3-AP and the result is verified. □

A *comb* is a graph obtained by adding a leaf to every vertex of a path. A *broken comb* is a connected subgraph of a comb that has a unique pair of leaves that realize the diameter. A unique pair of leaves that realize the diameter of a broken comb are called *antipodal vertices*. A *bijacent vertex* is a degree two vertex with two degree two neighbors. Observation 32 gives a reduction technique for computing anti-van der Waerden numbers of graphs with pendant vertices.

Observation 32. *[27] If a graph G has an exact r-coloring (with $r \geq 3$) that avoids rainbow 3-APs and has two leaves that are distance 2 from each other, then those leaves must be the same color. If u and v were leaves distance two from each other and $c(u) \neq c(v)$, then $\{u, x, v\}$ would be a rainbow 3-AP for any vertex x with $c(x) \neq c(u), c(v)$. This allows a graph to be reduced so that no vertex is adjacent to more than one leaf.*

Lemma 33. *[27] Let T be a broken comb with ℓ bijacent vertices and antipodal vertices u and v. If c is an exact r-coloring with no rainbow 3-APs such that $c(u)$ and $c(v)$ are both unique, then $r \leq \ell + 3$.*

Proof. Let T be a broken comb with ℓ bijacent vertices, antipodal vertices u and v, and exact r-coloring c with no rainbow 3-APs such that $c(u)$ and $c(v)$ are both unique. Consider the path from u to v and label the vertices $u = v_0, v_1, \ldots, v_d = v$ where $v_i v_{i+1} \in E(T)$ for $0 \leq i \leq d-1$.

Let v_i be a degree three vertex and, without loss of generality $i \leq \frac{d}{2}$. Let $w \in N(v_i)$ such that $\deg(w) = 1$. For all $x \in N(v_i) \setminus \{w\}$, either $\{w, u, x\}$ or $\{w, v, x\}$ is a 3-AP. Therefore, $c(x) = c(y)$ for all $x, y \in N(v_i)$. Also, $\{u, v_i, v_{2i}\}$ and $\{u, w, v_{2i}\}$ are 3-APs. Therefore, $c(v_i) = c(v_{2i}) = c(w)$ and v_i is the same color as all of its neighbors.

Let $\alpha \in c(T) \setminus \{c(v_1), c(u), c(v)\}$ and v_j be the smallest j such that $c(v_j) = \alpha$. This implies that $\deg(v_{j-1}), \deg(v_j) = 2$, because if either had degree 3, then $c(v_{j-1}) = c(v_j)$ and this contradicts that j was the smallest such index. If $\deg(v_{j+1}) = 3$, then $c(v_j) = c(v_{j+1})$ and either $\{u, v_{j/2}, v_j\}$ or $\{u, v_{(j+1)/2}, v_{j+1}\}$ is a rainbow 3-AP. Also, if $\deg(v_{j+1}) = 1$, then $j = d - 1$ and either $\{u, v_{j/2}, v_j\}$ or $\{u, v_{(j+1)/2}, v_{j+1}\}$ is a rainbow 3-AP. Therefore, $\deg(v_{j+1}) = 2$ which implies that v_j is a bijacent vertex. Therefore, there are at most $\ell + 3$ colors. $\square$

Theorem 34. *[27] If T is a tree with at most ℓ bijacent vertices in every path, then* $\operatorname{aw}(T, 3) \leq \ell + 4$.

Proof. Let T be a tree with at most ℓ bijacent vertices in every path and c be an exact r-coloring with no rainbow 3-APs. By Lemma 28, there exists a path that contains every color. Let $u = v_0, v_1, \ldots, v_d = v$ be a smallest path that contains all of the colors. Note that this implies $c(u)$ and $c(v)$ are unique colors on the path.

Let T' be the subgraph of T created by adding a leaf to each v_i that has degree at least 3 in T, for $2 \leq i \leq d - 2$. The subtree T' is a broken comb with at most ℓ bijacent vertices. By Lemma 33, T', has at most $\ell + 3$ colors. By Proposition 16, this implies $\operatorname{aw}(T, 3)$ has at most $\ell + 3$ colors. Hence, $\operatorname{aw}(T, 3) \leq \ell + 4$. $\square$

Corollary 35. *If the degree two vertices of a tree T induce a set of independent vertices and copies of K_2, then* $\operatorname{aw}(T, 3) \leq 4$.

The authors of Schulte et al. ([27]) believe Theorem 34 can be improved. In particular, there is an upper bound similar to the bound on paths in Theorem 12, which is a logarithmic function of the number of bijacent vertices contained in the tree.

Conjecture 36. *[27] There exists a constant C such that if T is a tree with at most ℓ bijacent vertices in every path, then* $\operatorname{aw}(T, 3) \leq \log_3(\ell) + C$.

If a trees diameter is odd, there exists an extremal 3-coloring that avoids rainbow 3-APs as seen in Proposition 37.

Proposition 37. *[27] If T is a tree such that* $\operatorname{diam}(T)$ *is odd, then* $4 \leq \operatorname{aw}(T, 3)$.

Proof. Let T be a tree such that $\mathrm{diam}(T) = d$ is odd. Let u and v be vertices in T such that $\mathrm{d}(u, v) = d$. Let c be the exact 3-coloring of T described. First, let $c(u) = blue$ and $c(v) = green$. For $w \in V(T) \setminus \{u, v\}$, define

$$c(w) := \begin{cases} green & \text{if } \mathrm{d}(u,w) = d \\ blue & \text{if } \mathrm{d}(v,w) = d \\ red & \text{otherwise.} \end{cases}$$

Note that all vertices with eccentricity d are colored *blue* or *green*. Furthermore, given a pair of vertices such that one is colored *blue* and the other is colored *green*, their distance must be d. Since d is odd, there is no rainbow 3-AP such that the middle vertex is colored *red*. Therefore, without loss of generality, a rainbow 3-AP must have the first vertex colored *blue*, the second vertex colored *green*, and the third vertex colored *red*. However, this would imply a *red* vertex has eccentricity d, which is a contradiction. Therefore, c has no rainbow 3-APs and thus $4 \leq \mathrm{aw}(T, 3)$. □

Corollary 38. *[27] If T is a tree with odd diameter and no bijacent vertices, then* $\mathrm{aw}(T, 3) = 4$.

3.4 Graph Products

In this section, the anti-van der Waerden number of the Cartesian product of graphs is investigated. Rehm et al. determined an upper bound of 4 for all Cartesian products of graphs.

Let G be a graph with $|V(G)| = n$ and H be a graph with $|V(H)| = m$. The *Cartesian product* of two graphs, denoted $G \square H$, is the graph where $V(G \square H) = V(G) \times V(H)$ and $(u, v)(x, y) \in E(G \square H)$ if and only if $u = x$ and $vy \in E(H)$, or $ux \in E(G)$ and $v = y$.

It is convenient to consider $G \square H$ with m copies of G, labeled $G_1, G_2, \ldots, G_m$, where G_i is the subgraph induced by the vertices of $G \square H$ of the form (u, i) for each $i \in V(H)$.

Observation 39. *The vertex set of $P_m \square P_n$ is given by $\{v_{i,j} \,|\, 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$. Further, $v_{i,j}$ can be found at the intersection of the ith row and jth column of $P_m \square P_n$. This convention allows for the computation of distances in these graphs based on the subscripts of the vertices. In particular, if $v_{i,j}$ and $v_{\ell,k}$ are in $P_m \square P_n$ then $\mathrm{d}(v_{i,j}, v_{\ell,k}) = |i - \ell| + |j - k|$.*

Lemma 40. *[27] Let G be a connected graph on n vertices and H be a connected graph on m vertices. Let c be a coloring of $G\square H$ with no rainbow 3-APs. If $G_1, G_2, \ldots, G_m$ are the labeled copies of G in $G\square H$, then $|c(G_j) \setminus c(G_i)| \leq 1$ for all $1 \leq i, j \leq m$.*

Proof. This proof begins with the special case where $G = P_m$ and $H = P_n$. The general case is proved by reducing it to the special case. Let c be a coloring of $G\square H$ with no rainbow 3-APs.

Case 1: $G = P_m$ and $H = P_n$.

Assume, for the sake of contradiction, that $|c(G_j) \setminus c(G_i)| > 1$. Without loss of generality, suppose *red* and *blue* appear in G_j but not G_i. Note $i \neq j$, otherwise $|c(G_j) \setminus c(G_i)| = 0$. Let $(v, j), (w, j) \in G_j$ such that $c((v, j)) = red$ and $c((w, j)) = blue$ and $v < w$. If $|j - i| = 1$, then $\{(v, j), (w, j), (w, i)\}$ is a rainbow 3-AP, a contradiction. Therefore, if $|j - i| = 1$, then $|c(G_j) \setminus c(G_i)| \leq 1$.

Now suppose, by way of mathematical induction, that $|c(G_j) \setminus c(G_i)| \leq 1$ whenever $|j - i| < k$. Suppose $|c(G_j) \setminus c(G_i)| > 1$ and $|j - i| = k$. Define $\mathrm{d}_G(v, w) = \ell$ and consider the isometric subgraph with corners $(v, i), (w, i), (v, j)$, and (w, j). If ℓ is even, then $\{(v, j), (v + \frac{\ell}{2}, i), (w, j)\}$ is a rainbow 3-AP since $c((v + \frac{\ell}{2}, i)) \neq red, blue$, hence ℓ is odd.

Let $(v + 1, j)$ and $(w - 1, j)$ be the vertices between (v, j) and (w, j) on a shortest path in G_j such that (v, j) is adjacent to $(v + 1, j)$ and (w, j) is adjacent to $(w - 1, j)$ in $G\square H$. This then implies $c((v + 1, j - 1)), c((w - 1, j - 1)) \in \{red, blue\}$ by the respective 3-APs $\{(v, j), (w, j), (v + 1, j - 1)\}$ and $\{(w, j), (v, j), (w - 1, j - 1)\}$. Moreover, $c((v + 1, j - 1)) = red$ by the 3-AP $\{(v, j), (v, i), (v + 1, j - 1)\}$ and $c((w - 1, j - 1)) = blue$ by the 3-AP $\{(w, j), (w, i), (w - 1, j - 1)\}$ since no vertex in G_i is *red* or *blue*. Therefore, $\{red, blue\} \subseteq c(V(G_{j-1}))$. However, notice that $|c(G_{j-1}) \setminus c(G_i)| \leq 1$ by the induction hypothesis since $|j - 1 - i| < k$. A contradiction since *red* and *blue* are in $c(G_{j-1})$ but not $c(G_i)$. Thus, $|c(G_j) \setminus c(G_i)| \leq 1$ for all $1 \leq i, j \leq m$.

Case 2: Without loss of generality $G \neq P_n$.

Let G_i and G_j be two of the labeled copies of G with $1 \leq i, j \leq m$ and assume $|c(G_j) \setminus c(G_i)| > 1$.

Without loss of generality, suppose *red* and *blue* appear in $c(G_j)$ but not in $c(G_i)$. Let $c((v,j)) = red$ and $c((w,j)) = blue$. Let P be a shortest path between v and w in G and let P' be a shortest path between i and j in H.

Consider the isometric subgraph formed by $P\square P'$. Let P_i and P_j be the labeled copies of P from G_i and G_j respectively. Notice $|c(P_j) \setminus c(P_i)| > 1$ and this is again case 1 which implies $P\square P'$ has a rainbow 3-AP. By Proposition 16, since $P\square P'$ has a rainbow 3-AP, $G\square H$ also has a rainbow 3-AP, a contradiction. □

Prior to the result of Theorem 59, the following results were preliminary bounds for $\mathrm{aw}(G\square H, 3)$.

Corollary 41. *[27] For all graphs G,*

$$\mathrm{aw}(G\square P_2, 3) \leq \mathrm{aw}(G, 3).$$

Proof. Consider $G\square P_2$ as two copies of G, labeled G_1 and G_2. Let c be a coloring of $G\square P_2$ such that there are no rainbow 3-APs. Since there are no rainbow 3-APs, we know that G_1 has at most $\mathrm{aw}(G,3) - 1$ colors used on it by Proposition 16. By Lemma 40 there can be at most 1 new color used on G_2, therefore $\mathrm{aw}(G\square P_2, 3) \leq \mathrm{aw}(G, 3)$. □

Proposition 42 relies on Lemma 40 and establishes an upper bound for the anti-van der Waerden number of graph products.

Proposition 42. *[27] If G is a connected graph on n vertices and H is a connected graph on m vertices, then*

$$\mathrm{aw}(G\square H, 3) \leq \min\{\mathrm{aw}(G,3) + m - 1, \mathrm{aw}(H,3) + n - 1\}.$$

Proof. Let c be a coloring of $G\square H$ with no rainbow 3-APs. Since there are no rainbow 3-APs, $|c(G_1)| \leq \mathrm{aw}(G,3) - 1$ by Proposition 16. By Lemma 40, each G_i has at most one color that was not used in $c(G_1)$. Therefore, there are at most $m - 1$ additional colors used in $G\square H$. Thus, $\mathrm{aw}(G\square H, 3) \leq \mathrm{aw}(G,3) + m - 1$. By a similar argument, $\mathrm{aw}(G\square H, 3) \leq \mathrm{aw}(H,3) + n - 1$. Thus, $\mathrm{aw}(G\square H, 3) \leq \min\{\mathrm{aw}(G,3) + m - 1, \mathrm{aw}(H,3) + n - 1\}$. □

The following colorings of $G\Box H$ are used in Proposition 43, Lemma 44, and Theorem 45.

Let c be a coloring of $G\Box H$ with no rainbow 3-APs. Suppose $G\Box H$ is labeled such that $|c(G_1)| \geq |c(G_i)|$ for $1 \leq i \leq m$. Let c' be the following coloring of H:

$$c'(G_i) = \begin{cases} \alpha : & c(G_i) \subseteq c(G_1), \\ c(G_i) \setminus c(G_1) : & c(G_i) \not\subseteq c(G_1). \end{cases}$$

Note $\alpha \notin c(G\Box H)$ and by Proposition 40, c' is well defined. Also,

$$|c(G\Box H)| = |c(G_1)| + |c'(H)| - 1 \tag{3.1}$$

since any vertex of $G\Box H$ is either colored with a color from $c(G_1)$ or it is the unique color in $c(G_i)$ that differs from $c(G_1)$.

If instead $G\Box H$ is labeled such that $|c(H_1)| \geq |c(H_i)|$ for $1 \leq i \leq n$, then let c'' be the following coloring of G:

$$c''(H_i) = \begin{cases} \beta : & c(H_i) \subseteq c(H_1), \\ c(H_i) \setminus c(H_1) : & c(H_i) \not\subseteq c(H_1). \end{cases}$$

Note $\beta \notin c(G\Box H)$ and by Proposition 40, c'' is well defined. Also,

$$|c(G\Box H)| = |c(H_1)| + |c''(G)| - 1 \tag{3.2}$$

since any vertex of $G\Box H$ is either colored with a color from $c(H_1)$ or it is the unique color in $c(H_i)$ that differs from $c(H_1)$.

Proposition 43 and Lemma 44 are useful for proving Theorem 45.

Proposition 43. *If for any two vertices u and w in G there exists a vertex v such that $\{u, v, w\}$ is a 3-AP in G, then*

$$\text{aw}(G\Box H, 3) \leq \text{aw}(G, 3) + \text{aw}(H, 3) - 2.$$

Proof. Suppose $c'(H)$ has a rainbow $3-$AP, $\{x, y, z\}$ in H and consider the following cases.

Case 1: $c'(y) \neq \alpha$.

Suppose without loss of generality $c'(x) = \textit{red}$ and $c'(y) = \textit{blue}$. There exists a vertex $(u, x) \in$

$G\Box H$ such that $c((u,x)) = red$ and a vertex $(v,y) \in G\Box H$ such that $c((v,y)) = blue$. Form the rainbow 3-AP $\{(u,x),(v,y),(u,z)\}$. Since $c'(z) \neq red, blue$ then neither *red* nor *blue* is used on any vertex of G_z, therefore this forms a rainbow 3-AP.

Case 2: $c'(y) = \alpha$.

Suppose without loss of generality $c'(x) = red$ and $c'(z) = blue$. There exists vertices (u,x) and $(w,z) \in G\Box H$ such that $c((u,x)) = red$ and $c((w,z)) = blue$. Since u and w are in G there exists a vertex v such that $\{u,v,w\}$ is a 3-AP in G. Therefore, choose $(v,y) \in G_y$ and since $c'(y) = \alpha$, then $c((v,y)) \neq red, blue$. Thus $\{(u,x),(v,y),(w,z)\}$ forms a rainbow 3-AP in $G\Box H$.

Therefore, if $c'(H)$ has a rainbow 3-AP then $c(G\Box H)$ has a rainbow 3-AP. To guarantee $c'(H)$ has no rainbow 3-AP, $|c'(H)| \leq \mathrm{aw}(H,3) - 1$. Similarly, to guarantee $c(G_1)$ has no rainbow 3-AP, $|c(G_1)| \leq \mathrm{aw}(G,3) - 1$. By Equation 3.1

$$\begin{aligned} |c(G\Box H)| &= |c(G_1)| + |c'(H)| - 1 \\ &\leq \mathrm{aw}(G,3) - 1 + \mathrm{aw}(H,3) - 1 - 1 \\ &= \mathrm{aw}(G,3) + \mathrm{aw}(H,3) - 3. \end{aligned}$$

Thus if $\mathrm{aw}(G,3) + \mathrm{aw}(H,3) - 2$ colors are used on $G\Box H$ then $c(G\Box H)$ will contain a rainbow 3-AP.

The analogous result holds if for any two vertices x and z in H there exists a vertex y such that $\{x,y,z\}$ is a 3-AP in H, then by the same arguments $\mathrm{aw}(G\Box H,3) \leq \mathrm{aw}(G,3) + \mathrm{aw}(H,3) - 2$. $\Box$

Lemma 44. *If $c'(H)$ has a rainbow 3-AP, $\{x,y,z\}$ such that $c'(x) = red, c'(y) = \alpha$, and $c'(z) = blue$, then $|c(G\Box H)| \leq m+1$.*

Similarly, if $c''(G)$ has a rainbow 3-AP, $\{u,v,w\}$ such that $c''(u) = red, c''(v) = \beta$, and $c''(w) = blue$, then $|c(G\Box H)| \leq n+1$.

Proof. Since $\{x,y,z\}$ is a rainbow 3-AP in H there exists vertices $(u,x) \in G\Box H$ and $(w,z) \in G\Box H$ such that $c((u,x)) = red$ and $c((w,z)) = blue$. Let P be the shortest path from u to w in G. Consider the 3-AP $\{(w,x),(w,y),(w,z)\}$ in $G\Box H$, $c((w,z)) = blue$ and $c((w,y)) \neq blue$ and

$c((w,x)) \neq blue$. If $c((w,y))$ was $blue$ then $c'(y) = blue$ since $blue \notin c(G_1)$. If $c((w,x)) = blue$, then $|c(G_1) \setminus c(G_x)| \geq 2$, a contradiction to Lemma 40. Also, $c((w,x)) \neq red$ otherwise $\{(w,x),(w,y),(w,z)\}$ forms a rainbow 3-AP in $G \square H$ since $c((w,y)) \neq red$.

Therefore $c((w,x)) = c((w,y)) = \gamma$ for some color γ where $\gamma \neq blue, red$. Moreover, $c((v,y)) = \gamma$ for all $(v,y) \in G_y$, if not there exists a vertex (v,y) such that $c((v,y)) \neq \gamma$. However, this implies $\{(w,x),(v,y),(w,z)\}$ forms a rainbow 3-AP, a contradiction. Therefore G_y is monochromatic under c. By Lemma 40, each G_i can differ from G_y by at most 1 color, therefore $|c(G_i)| \leq 2$ for all i and one of those two colors must be γ. Now, consider the following cases.

Case 1: $|c(G_1)| = 1$.

Since $|c(G_1)| = 1$, G_i is monochromatic for all i, therefore m colors can be used. However, since $c'(y) = \alpha$, then $c(G_y) \subseteq c(G_1)$. Therefore at most $m-1$ colors have been used on $G \square H$.

Case 2: $|c(G_1)| = 2$.

Since each G_i can introduce at most 1 new color there are at most $2+m-1$ colors. However, since $c'(y) = \alpha$, $c(G_y) \subseteq c(G_1)$. Therefore at most $2+m-2$ colors have been used on $G \square H$.

Thus, if $m+1$ colors are used on $G \square H$, then there is a rainbow 3-AP. Therefore $\operatorname{aw}(G \square H, 3) \leq m+1$. A similar argument can be used with c'' to show $\operatorname{aw}(G \square H, 3) \leq n+1$. $\square$

The following theorem (Theorem 45) relies on the results of Proposition 43 and Lemma 44 to find an upper bound for $\operatorname{aw}(G \square H, 3)$. This upper bound was an initial bound and was later improved by Rehm, Schulte, and Warnberg as seen in Theorem 59.

Theorem 45. *For connected graphs G and H such that $|G| = n$ and $|H| = m$,*

$$\operatorname{aw}(G \square H, 3) \leq \min\{\max\{\operatorname{aw}(G,3)+\operatorname{aw}(H,3)-2, n+1\}, \max\{\operatorname{aw}(G,3)+\operatorname{aw}(H,3)-2, m+1\}\}.$$

Proof. This proof is broken up into four cases.

Case 1: $c'(G\Box H)$ does not have a rainbow 3-AP of the form $red, \alpha, blue$.

By Case 1 of Proposition 43 $\mathrm{aw}(G\Box H, 3) \leq \mathrm{aw}(G,3) + \mathrm{aw}(H,3) - 2$.

Case 2: $c(G\Box H)$ has a 3-AP of the form $red, \alpha, blue$.

If for all $u, w \in V(G)$ there exists a vertex $v \in V(G)$ such that $\{u, v, w\}$ is a 3-AP, then by Proposition 43 $\mathrm{aw}(G\Box H, 3) \leq \mathrm{aw}(G,3) + \mathrm{aw}(H,3) - 2$. If G does not have that property, then by Lemma 44 $\mathrm{aw}(G\Box H, 3) \leq m + 1$.

Case 3: $c''(G\Box H)$ does not have a rainbow 3-AP of the form $red, \beta, blue$.

By Case 1 of Proposition 43 $\mathrm{aw}(G\Box H, 3) \leq \mathrm{aw}(G,3) + \mathrm{aw}(H,3) - 2$.

Case 4: $c''(H)$ has a 3-AP of the form $red, \beta, blue$.

If for all $u, w \in V(H)$ there exists a vertex $v \in V(H)$ such that $\{u, v, w\}$ is a 3-AP, then by Proposition 43 $\mathrm{aw}(G\Box H, 3) \leq \mathrm{aw}(G,3) + \mathrm{aw}(H,3) - 2$. If H does not have that property, then by Lemma 44 $\mathrm{aw}(G\Box H, 3) \leq n + 1$. □

After determining upper bounds for the anti-van der Waerden number of Cartesian products of graphs using the anti-van der Waerden of each graph in the product, an exploration of path graphs and their Cartesian products began. The motivation for this arose from the proof of Lemma 40 when the result was more easily proved for $P_m \Box P_n$ and then extended to general graphs $G\Box H$. Recall by Observation 39 that the Cartesian product of two paths will be thought of as a grid system with vertex $(1, 1)$ in the upper left corner and vertex (m, n) in the lower right.

Lemma 46. *[22] Let $G = P_m \Box P_n$ and c be an exact r-coloring of G with $r \geq 3$ that avoids rainbow 3-APs. If $c(v_{i,j}) = red$ and $c(v_{i-1,j+1}) = blue$, then $c(v_{k,\ell}) \in \{red, blue\}$ when $k \geq i$ and $\ell \geq j+1$ or $k \leq i-1$ and $\ell \leq j$. Further, if $c(v_{i,j}) = red$ and $c(v_{i-1,j-1}) = blue$, then $c(v_{k,\ell}) \in \{red, blue\}$ when $k \geq i$ and $\ell \leq j-1$ or $k \leq i-1$ and $\ell \geq j$.*

Proof. Consider the case when $c(v_{i,j}) = red$ and $c(v_{i-1,j+1}) = blue$ (see Figure 3.2). Define $v_{k,\ell}$ so that $k \geq i$ and $\ell \geq j+1$. Notice that $\mathrm{d}(v_{k,\ell}, v_{i,j}) = \mathrm{d}(v_{k,\ell}, v_{i-1,j+1}) = k - i + \ell - j$. This means

$\{v_{i,j}, v_{k,\ell}, v_{i-1,j+1}\}$ is a 3-AP and since c avoids rainbow 3-APs $c(v_{k,\ell}) \in \{red, blue\}$. A similar argument can be made in the other three situations. $\square$

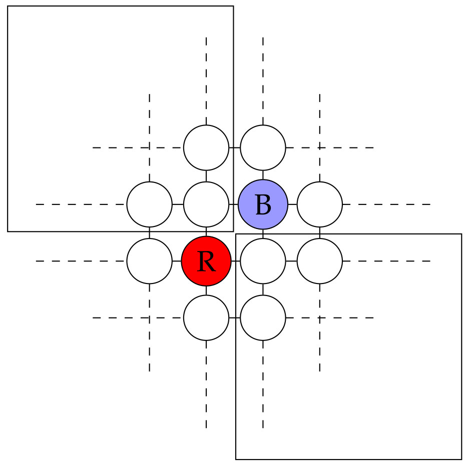

Figure 3.2 Vertex R (or $v_{i,j}$) is *red* and vertex B (or $v_{i-1,j+1}$) is *blue* force the Northwest and Southeast blocks to be *red* or *blue*.

Lemma 47. *[22] Let $G = P_m \square P_n$ and c be an exact r-coloring of G that avoids rainbow 3-APs with $r \geq 3$. If $c(v_{i,k}) = \{red\}$ for fixed i and $1 \leq k \leq n$, $S_1 = \{v_{s,t} \,|\, 1 \leq s < i, 1 \leq t \leq n\}$ and $S_2 = \{v_{s,t} \,|\, i < s \leq m, 1 \leq t \leq n\}$, then $|c(S_i) \cup \{red\}| \leq 2$ for $1 \leq i \leq 2$.*

Proof. Assume, without loss of generality, that $c(v_{\ell,j}) = blue$ for some j and $i < \ell \leq m$ and rows $i+1$ to $\ell - 1$ are monochromatic *red*. By Lemma 46, if $c(v_{s,t}) = green$ for $\ell \leq s \leq m$, $1 \leq t \leq n$ and $t \neq j$, then either $\{v_{\ell,j}, v_{s,t}, v_{\ell-1,j-1}\}$ or $\{v_{\ell,j}, v_{s,t}, v_{\ell-1,j+1}\}$ is rainbow. This implies that for $t \neq j$, $c(v_{s,t}) \in \{red, blue\}$. However, using Lemma 46 with $v_{s,j}$, one of $\{v_{s,j}, v_{\ell,j}, v_{s-1,j-1}\}$, $\{v_{s,j}, v_{\ell,j}, v_{s-1,j+1}\}$, $\{v_{s,j}, v_{\ell-1,j}, v_{s-1,j-1}\}$ or $\{v_{s,j}, v_{\ell-1,j}, v_{s-1,j+1}\}$ is a rainbow 3-AP in G. Thus, no such $v_{s,t}$ is *green*. A similar argument applies when $1 \leq \ell < i$ and rows $\ell + 1$ to $i - 1$ are monochromatic *red*. $\square$

Lemma 47 says that if there is a monochromatic row in some $P_m \square P_n$, then at most one new color can be introduced below the monochromatic row and at most one new color can be introduced above the monochromatic row. Note that the argument can be easily applied to monochromatic columns. Corollary 48 states this result.

Corollary 48. *[22] Let $G = P_m \square P_n$ and c be an exact r-coloring of G such that c avoids rainbow 3-APs and $r \geq 3$. If $c(v_{i,k}) = \{red\}$ for fixed k and $1 \leq i \leq m$, $S_1 = \{v_{s,t} \,|1 \leq s \leq m,\, 1 \leq t < k\}$ and $S_2 = \{v_{s,t} \,|\, 1 \leq s \leq m,\, k < t \leq n\}$, then $|c(S_i) \cup \{red\}| \leq 2$.*

Lemma 49 will be useful in combination with Proposition 16 in determining the anti-van der Waerden number. In particular, Lemma 49 establishes the possible 3-colorings of $P_2 \square P_{2\ell+1}$ that avoid rainbow 3-APs. These colorings are achieved when two non-adjacent corners are given unique colors and the remainder of the graph is colored with the third color.

Lemma 49. *[22] If $G = P_2 \square P_{2\ell+1}$ and $\ell \geq 1$, then there are precisely two exact 3-colorings of G that avoid rainbow 3-APs.*

Proof. Let c be an exact 3-coloring of G that avoids rainbow 3-APs. Without loss of generality, let $c(v_{1,1}) = red$. If $c(v_{2,1}) = red$, then by Corollary 48, G is colored with at most two colors. Thus, $c(v_{2,1}) = blue$. If $c(v_{1,2}) = green$, then $\{v_{1,2}, v_{1,1}, v_{2,1}\}$ is a rainbow 3-AP. Now, consider the following cases.

Case 1: $c(v_{1,2}) = red$.

By Lemma 46, $c(v_{2,j}) \in \{red, blue\}$ for $2 \leq j \leq 2\ell + 1$. If $c(v_{2,2}) = blue$, then Lemma 46 forces both the top and bottom rows to be colored *red* or *blue* contradicting that c was an exact 3-coloring. Thus, $c(v_{2,2})$ must be *red*. By Corollary 48, columns 3 through $2\ell + 1$ must be *red* and *green*, but the bottom row is also *red* and *blue*; thus, $c(v_{2,j}) = red$ for $3 \leq j \leq 2\ell + 1$. This means $c(v_{1,i}) = green$ for some $3 \leq i \leq 2\ell + 1$. If $i \neq 2\ell + 1$, then $\{v_{1,i}, v_{2,1}, v_{2,i+1}\}$ is a rainbow 3-AP. Thus, for $i < 2\ell + 1$, $c(v_{1,i}) = red$ and $c(v_{1,2\ell+1}) = green$. This is an exact 3-coloring that avoids rainbow 3-APs.

Case 2: $c(v_{1,2}) = blue$.

If $c(v_{2,2}) = green$, then there exists an obvious rainbow 3-AP. If $c(v_{2,2}) = blue$, apply an argument similar to Case 1 and achieve a symmetric coloring. Consider if $c(v_{2,2}) = red$. Let $c(v_{i,j}) = green$ such that j is minimal. By Corollary 48, column j cannot be monochromatic since *red* and *blue* appear in column 1. If $c(v_{1,j}) = red$ and $c(v_{2,j}) = green$, then $c(v_{1,j-1}) \neq green$ by minimality of j, $c(v_{1,j-1}) \neq blue$ by the rainbow 3-AP $\{v_{1,j-1}, v_{2,2}, v_{2,j}\}$ and $c(v_{1,j-1}) \neq red$ by the rainbow 3-AP $\{v_{1,j-1}, v_{2,1}, v_{2,j}, \}$. If $c(v_{1,j}) = blue$ and $c(v_{2,j}) = green$, a similar argument can be made. Finally, the symmetry of columns 1 and 2 demonstrate that $c(v_{1,j}) \neq green$.

Therefore, there are two exact 3-colorings on G that avoid rainbow 3-APs. □

A careful analysis of the anti-van der Waerden number for both $P_2 \square P_n$ and $P_3 \square P_n$ lead to many isometric subgraphs that help determine a bound for the anti-van der Waerden number of $G \square H$ for connected graphs G and H.

Many of the results for the anti-van der Waerden number of $G \square H$ require an arbitrary coloring of a graph. Lemma 49 allows the elimination of one (or more) colors from half of the graph under the right circumstances.

Proposition 50. *[22] For every $\ell \geq 1$,*

$$\mathrm{aw}(P_2 \square P_{2\ell}, 3) = 3.$$

Proof. Consider the graph $P_2 \square P_{2\ell}$ and proceed by induction on ℓ. If $\ell = 1$, then By Theorem 13 $\mathrm{aw}(P_2 \square P_2, 3) = \mathrm{aw}(C_4, 3) = 3$.

For the inductive hypothesis, assume that $\mathrm{aw}(P_2 \square P_{2\ell}, 3) = 3$. Now consider $P_2 \square P_{2\ell+2}$ with an exact 3-coloring c which avoids rainbow 3-APs. The graph $P_2 \square P_{2\ell+2}$ can be thought of as the union of the two isometric subgraphs formed by columns 1 through 3 and columns 3 through $2\ell + 2$. More technically, let $G_1 = P_2 \square P_3$ with $V(G_1) = \{v_{1,1}, v_{1,2}, v_{1,3}, v_{2,1}, v_{2,2}, v_{2,3}\}$ and let $G_2 = P_2 \square P_{2\ell}$ with $V(G_2) = \{v_{1,3}, v_{2,3}, v_{1,4}, v_{2,4}, \ldots, v_{1,2\ell+2}, v_{2,2\ell+2}\}$. Then $V(G_1) \cap V(G_2) = \{v_{1,3}, v_{2,3}\}$ and $G_1 \cup G_2 = P_2 \square P_{2\ell+2}$ (see Figure 3.3). For the following cases, let c be an exact 3-coloring and let $S = \{v_{1,3}, v_{2,3}\}$.

Case 1: $|c(S)| = 2$.

Without loss of generality, let $c(v_{1,3}) = blue$ and $c(v_{2,3}) = red$. By the inductive hypothesis, a third color cannot be introduced into G_2 such that there is no rainbow 3-AP. However, by Lemma 49, there exists a unique exact 3-coloring that avoids rainbow 3-AP's in G_1. Without loss of generality, consider the following coloring of G_1, $c(v_{1,1}) = green$, and all other vertices in G_1 are colored *blue* (see Figure 3.3).

Now, focusing on the vertex pairs $v_{1,1}$, $v_{2,2}$ and $v_{1,2}$, $v_{2,3}$, Lemma 46 forces $c(v_{1,j}) = blue$ for $2 \leq j \leq 2\ell + 2$. This however yields the rainbow 3-AP, $\{v_{1,4}, v_{1,1}, v_{2,3}\}$, a contradiction.

Case 2: $|c(S)| = 1$

Without loss of generality let $c(S) = \{red\}$. By Lemma 49 at most one new color can be added to G_1 and by the induction hypothesis at most one new color can be added to G_2 while still avoiding rainbow 3-APs. Without loss of generality, assume the color introduced in G_1 is *blue* and the color introduced in G_2 is *green*. If $c(v_{1,2}) = blue$ then, by Lemma 46, $c(v_{1,j}) = red$ for $3 \leq j \leq 2\ell + 2$. Now if $c(v_{2,\ell}) = green$ for some $4 \leq \ell \leq 2\ell + 2$, then Lemma 46 says that $c(v_{2,1}) = red$, but then $\{v_{2,1}, v_{2,\ell}, v_{1,2}\}$ is a rainbow 3-AP. A similar argument can be made if $c(v_{2,2}) = blue$, so $c(v_{1,2}) = c(v_{2,2}) = red$.
Now let $c(v_{1,1}) = blue$, then by Lemma 46 $c(v_{1,j}) = red$ for $4 \leq j \leq 2\ell + 2$. This forces $c(v_{2,\ell}) = green$ for some $4 \leq \ell \leq 2\ell + 2$. If $4 \leq \ell \leq 2\ell + 1$ then $\{v_{2,\ell}, v_{1,1}, v_{1,\ell+1}\}$ is a rainbow 3-AP. If $\ell = 2\ell + 2$, then $\{v_{1,1}, v_{1,k+2}, v_{2,2\ell+2}\}$ is a rainbow 3-AP. Similarly, $c(v_{2,1}) \neq blue$ which means $|c(G_1)| = 1$. This in turn implies that $|c(G_2)| = 3$ which, as noted earlier, has a rainbow 3-AP via the inductive hypothesis.
It has been demonstrated that every exact 3-coloring of $P_2 \square P_{2\ell+2}$ will result in a rainbow 3-AP. Thus, $\text{aw}(P_2 \square P_{2\ell}, 3) = 3$ for all $\ell \geq 1$. $\square$

Lemma 51. *[22] If $G = P_m \square P_n$ and $m + n = 2\ell + 1$ for some $\ell \geq 1$, then*

$$4 \leq \text{aw}(G, 3).$$

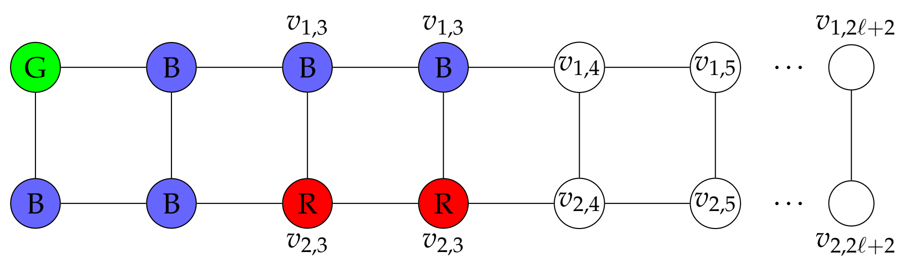

Figure 3.3 Note the identification of vertices $v_{1,3}$ and $v_{2,3}$ implies the figure shows $P_2\square P_3 \cup P_2\square P_{2\ell} = P_2\square P_{2\ell+2}$.

Proof. Consider the exact 3-coloring c where $c(v_{1,1}) = red$, $c(v_{m,n}) = blue$ and the remaining vertices are *green*. Note $\mathrm{d}(v_{1,1}, v_{m,n}) = m+n-2$ which, by assumption, is odd so there does not exist a vertex equidistant from both $v_{1,1}$ and $v_{m,n}$, i.e. there is no 3-AP of the form $\{v_{1,1}, v_{i,j}, v_{m,n}\}$. This means if a rainbow 3-AP exists it must be of the form $\{v_{1,1}, v_{m,n}, v_{i,j}\}$ (or similarly $\{v_{m,n}, v_{1,1}, v_{i,j}\}$) which implies there is some vertex $v_{i,j}$ that is distance $m+n-2$ from $v_{1,1}$ or $v_{m,n}$. However, this cannot happen since $v_{1,1}$ and $v_{m,n}$ are, the only two vertices distance $m+n-2$ apart from one another. Thus, an exact 3-coloring that avoids rainbow 3-APs has been constructed, therefore $4 \leq aw(G,3)$. □

Proposition 52. *[22] For every* $\ell \geq 1$,

$$\mathrm{aw}(P_2\square P_{2\ell+1}, 3) = 4.$$

Proof. Let $G = P_2\square P_{2\ell+1}$. First notice that $4 \leq \mathrm{aw}(G,3)$ by Lemma 49. Now, consider the two isometric subgraphs $G_1 = P_2\square P_2$ and $G_2 = P_2\square P_{2\ell}$ with $S = V(G_1)\cap V(G_2) = \{v_{1,2}, v_{2,2}\}$. Let c be an exact 4-coloring of G. Note that G_1 and G_2 must share at least one color. If $|c(G_1)| = 2$ and $|c(G_2)| = 2$ then at most three colors have been used. This implies $|c(G_i)| = 3$ for some $i \in \{1,2\}$, but $\mathrm{aw}(G_1,3) = \mathrm{aw}(G_2,3) = 3$ by Theorem 13 and Proposition 50, respectively. Thus, there exists a rainbow 3-AP in either G_1 or G_2. Therefore, $\mathrm{aw}(G,3) = 4$. □

Proposition 53. *[22] For every* $\ell \geq 1$,

$$aw(P_3\square P_{2\ell}, 3) = 4.$$

Proof. Consider the graph $G = P_3 \square P_{2\ell}$. Since $3 + 2\ell = 2(\ell + 1) + 1$, then $4 \leq \text{aw}(G, 3)$ by Lemma 51.

Let c be an exact 4-coloring of G. Now consider the two isometric subgraphs G_1 and G_2 each of which are $P_2 \square P_{2\ell}$ graphs where $V(G_1) \cap V(G_2) = \{v_{2,1}, v_{2,2}, \ldots, v_{2,2\ell}\}$. By Proposition 50, G_1 and G_2 must have at most two colors to avoid a rainbow 3-AP. This means the coloring c must give a rainbow 3-AP, thus $\text{aw}(G, 3) = 4$. □

Lemma 54. *[22] The anti-van der Waerden number of $P_3 \square P_3$ is 3, i.e.*

$$\text{aw}(P_3 \square P_3, 3) = 3.$$

Proof. Let $G = P_3 \square P_3$ and c be an exact 3-coloring of G. Consider the two isometric subgraphs G_1 and G_2 each of which are $P_2 \square P_3$ graphs. Let $S = V(G_1) \cap V(G_2) = \{v_{2,1}, v_{2,2}, v_{2,3}\}$. If each of these vertices is assigned a different color, then G clearly has a rainbow 3-AP. Now, consider the following cases.

Case 1: $|c(S)| = 1$.

Suppose, without loss of generality, that $c(S) = \{red\}$. By Lemma 47 neither G_1 nor G_2 can have three colors. Let $c(v_{1,1}) = blue$ and suppose $c(v_{3,j}) = green$ for some $j \in \{1, 2, 3\}$. Then, either $\{v_{1,1}, v_{2,1}, v_{3,1}\}$, $\{v_{3,2}, v_{1,1}, v_{2,3}\}$ or $\{v_{1,1}, v_{2,2}, v_{3,3}\}$ is a rainbow 3-AP. Therefore, $c(v_{1,1}) = red$ and by symmetry

$$c(v_{1,1}) = c(v_{1,3}) = c(v_{3,1}) = c(v_{3,3}) = red.$$

This leaves only $v_{1,2}$ and $v_{3,2}$ uncolored and assigning them the colors *blue* and *green* yields the rainbow 3-AP $\{v_{1,2}, v_{2,2}, v_{3,2}\}$.

Case 2: $|c(S)| = 2$.

Without loss of generality, let $c(S) = \{blue, green\}$. A coloring described in Lemma 49 indicates that if a third color is added to G_1 or G_2, without loss of generality, $c(\{v_{1,2}, v_{1,3}, v_{2,1}, v_{2,2}\}) = \{blue\}$, $c(v_{1,1}) = red$, and $c(v_{2,3}) = green$. If $c(v_{3,1}) = blue$, $c(v_{3,1}) = green$ or $c(v_{3,1}) = red$, then

$\{v_{1,1}, v_{2,3}, v_{3,1}\}$, $\{v_{1,1}, v_{2,1}, v_{3,1}\}$ or $\{v_{1,2}, v_{3,1}, v_{2,3}\}$ is a rainbow 3-AP, respectively.

Therefore, $\text{aw}(G, 3) = 3$. $\square$

Proposition 55. *[22] For every $\ell \geq 1$,*

$$\text{aw}(P_3 \square P_{2\ell+1}, 3) = 3.$$

Proof. First, if $k = 1$, then $\text{aw}(P_3 \square P_3, 3) = 3$ by Lemma 54. Assume by mathematical induction that $\text{aw}(P_3 \square P_{2\ell+1}, 3) = 3$ and now consider the graph $P_3 \square P_{2\ell+3}$. Let c be an exact 3-coloring of $P_3 \square P_{2\ell+3}$ and consider the two isometric subgraphs $G_1 = P_3 \square P_3$ and $G_2 = P_3 \square P_{2\ell+1}$ where $S = V(G_1) \cap V(G_2) = \{v_{1,3}, v_{2,3}, v_{3,3}\}$. Note that $\text{aw}(G_1, 3) = \text{aw}(G_2, 3) = 3$ by the base case and induction hypothesis, respectively. Notice that $|c(S)| \neq 2$, otherwise adding a third color to either G_1 or G_2 would yield a rainbow 3-AP. Clearly $|c(S)| \neq 3$, so suppose $|c(S)| = 1$. Without loss of generality, let $c(S) = \{red\}$, $c(V(G_1)) = \{red, blue\}$, and $c(V(G_2)) = \{red, green\}$.

If $c(v_{1,2}) = blue$, then $c(v_{1,j}) = red$ for $4 \leq j \leq 2\ell + 3$ by Lemma 46. If $c(v_{2,\ell}) = green$ for some $4 \leq \ell \leq 2\ell + 2$, then $\{v_{2,\ell}, v_{1,2}, v_{1,\ell+1}\}$ is a rainbow 3-AP so $c(v_{2,\ell}) = red$. If $c(v_{2,2\ell+3}) = green$, then $\{v_{1,2}, v_{1,k+3}, v_{2,2\ell+3}\}$ is a rainbow 3-AP. Thus, the color *green* must only appear in the third row. A similar argument, using the 3-AP $\{v_{3,\ell}, v_{1,2}, v_{2,\ell+1}\}$, shows that $c(v_{3,\ell}) = red$. If $c(v_{3,2\ell+3}) = green$, then $c(v_{2,1})$ must be *blue* since $\{v_{2,1}, v_{3,2\ell+3}, v_{1,2}\}$ is a 3-AP. However, this creates the rainbow 3-AP $\{v_{3,2\ell+3}, v_{2,1}, v_{1,2\ell+3}\}$. This implies $c(v_{1,2}) = red$ and by symmetry $c(v_{3,2}) = red$.

Now, if $c(v_{2,2}) = blue$, then by Lemma 46, a third color cannot be introduced in G_2. Therefore, all of column two is colored *red*.

If $c(v_{2,1}) = blue$, then by Lemma 46 a third color cannot be introduced in G_2. Thus $c(v_{2,1}) = red$. If $c(v_{1,1}) = blue$, then by Lemma 46 $c(v_{1,j}) = red$ for $4 \leq j \leq 2\ell + 3$. If $c(v_{2,\ell}) = green$ for some $4 \leq \ell \leq 2\ell + 2$, then $\{v_{1,\ell+1}, v_{1,1}, v_{2,\ell}\}$ is a rainbow 3-AP. If $c(v_{2,2\ell+3}) = green$, then by Lemma 46 $c(v_{3,1}) = red$ which yields the rainbow 3-AP $\{v_{1,1}, v_{2,2\ell+3}, v_{3,1}\}$. Thus, $c(v_{2,2\ell+3}) = red$. If $c(v_{3,\ell}) = green$ for $4 \leq \ell \leq 2\ell + 2$, then $\{v_{3,\ell}, v_{1,1}, v_{2,\ell+1}\}$ is a rainbow 3-AP. Therefore, $c(v_{3,\ell}) = red$. Finally, if $c(v_{3,2\ell+3}) = green$, then $\{v_{1,1}, v_{2,k+2}, v_{3,2\ell+3}\}$ is a rainbow 3-AP. Therefore, any 3-coloring of G yields a rainbow 3-AP. $\square$

It will be useful to find isometric graphs that have at least three colors. This is made possible by the following lemma.

Lemma 56. *[22] If G is a connected graph on at least three vertices with an exact r-coloring c where $r \geq 3$, then there exists a subgraph H in G with at least three colors where H is either an isometric path or $H = C_3$.*

Proof. Let G be a connected graph with at least 3 vertices and let c be an exact r-coloring of G with $r \geq 3$. Choose $u, v \in V(G)$ such that $uv \in E(G)$, $c(u) = red$ and $c(v) = blue$. Further, let $w \in V(G)$ such that $\mathrm{d}(v, w)$ is minimal and $c(w) = green$, P_v be a shortest path from v to w, and P_u be a shortest path from u to w. If $P_u \subseteq P_v$ or $P_v \subseteq P_u$, then one of P_u or P_v is H.
Now assume $P_u \not\subseteq P_v$ and $P_v \not\subseteq P_u$. If the lengths of P_u and P_v differ by two or more, then there is a contradiction about the minimality of P_u or P_v. If the lengths of P_u and P_v differ by one, then extending P_u to contain v or P_v to contain u gives H. If the lengths of P_u and P_v are the same, define $V(P_x) = \{x, x_1, x_2, \ldots, x_\ell, w\}$ for $x \in \{u, v\}$. If $c(u_i) = green$, then either $\mathrm{d}(v, w) = \mathrm{d}(v, u_i)$ or $\mathrm{d}(v, u_i) < \mathrm{d}(v, w)$. In the former case, the shortest path from v to u to u_i is H, and the latter case contradicts the minimality of P_v. If $c(u_i) \neq red$, then $P_u = H$ and if $c(v_i) \neq blue$, then $P_v = H$. Thus, the last situation to consider is if $c(u_i) = red$ and $c(v_i) = blue$ for all i. However, the subgraph of G induced by $\{u_\ell, w, v_\ell\}$ is H. □

Lemma 57. *[22] Assume G and H are connected graphs and consider the graph $G \square H$. Let $V(H) = \{v_1, v_2, \ldots, v_n\}$, $n \geq 3$, and suppose c is an exact r-coloring such that $r \geq 3$, c avoids rainbow 3-APs and $|c(V(G_i))| \leq 2$ for $1 \leq i \leq n$. If $v_i v_j \in E(H)$, then $|c(V(G_i) \cup V(G_j))| \leq 2$.*

Proof. If G_i is monochromatic and G_j is monochromatic then the result is immediate. If G_i is monochromatic and G_j is bichromatic then either $|c(V(G_i) \cup V(G_j))| \leq 2$ or $|c(V(G_j)) \setminus c(V(G_i))| = 2$. The former is the desired result and the latter contradicts Lemma 40. Now consider the case where at least one of G_i or G_j has three or more colors. Without loss of generality, assume G_i has three or more colors. Then there exists an C_3 subgraph or an isometric path with at least three colors, by Lemma 56, in G_i. If G_i has a C_3 subgraph with three colors, then there is an immediate rainbow 3-AP in $G \square H$. If G_i has an isometric path with at least three colors, let $\rho^{(i)}$ be the shortest such path G_i and $\rho^{(j)}$ be the corresponding path in G_j. This creates a $P_2 \square P_y$ where y is the length of $\rho^{(i)}$. By Lemma 49 and Proposition 50 there exists a rainbow 3-AP in $G \square H$.

Assume G_i and G_j each have two colors with $|c(V(G_i) \cup V(G_j))| \geq 3$. Since $|c(V(G_i)) \setminus c(V(G_j))| \leq 1$, by Lemma 40, then they must share a color. Without loss of generality, let $c(V(G_i)) = \{red, blue\}$ and $c(V(G_j)) = \{blue, green\}$. Pick a *red* vertex, say $v_{i,\alpha}$, in G_i with a *blue* neighbor, namely v. Also, choose $v_{j,\beta}$ in G_j such that $c(v_{j,\beta}) = green$. Let $v_{i,\beta}$ be the vertex in G_i that corresponds to $v_{j,\beta}$ and let $P^{(i)}$ be a shortest path from $v_{i,\alpha}$ to $v_{i,\beta}$ in G_i and $P^{(j)}$ be the corresponding path in G_j. Notice that $P^{(i)}$ and $P^{(j)}$ form an isometric $P_2 \square P_x$ in $G \square H$ where x is the length of $P^{(i)}$. If $P_2 \square P_x$ has no *blue* vertices, then $\{v, v_{i,\alpha}, v_{j,\alpha}\}$ is a rainbow 3-AP. If $P_2 \square P_x$ has a *blue* vertex and x is even, then there is a rainbow 3-AP since $\operatorname{aw}(P_2 \square P_{2\ell}, 3) = 3$ by Proposition 50. If x is odd, then by Lemma 49, so $c(v_{j,\alpha}) = c(v_{i,\beta}) = blue$. Now, extend to $P_2 \square P_x$ to include a corresponding path from G_k where $v_j v_k \in E(H)$, which gives a $P_3 \square P_x$ subgraph. If $P_3 \square P_x$ is an isometric subgraph of $G \square H$, then there is a rainbow 3-AP since $\operatorname{aw}(P_3 \square P_{2\ell+1}, 3) = 3$, by Proposition 55. If $P_3 \square P_x$ is not an isometric subgraph of $G \square H$, then it must correspond to an isometric subgraph $C_3 \square P_x$ of $G \square H$. Let $v_{k,\beta}$ be the vertex in G_k that corresponds to $v_{j,\beta}$. However, $c(v_{k,\beta})$ cannot be *red*, *blue*, or *green* due to 3-APs $\{v_{i,\beta}, v_{j,\beta}, v_{k,\beta}\}$, $\{v_{k,\beta}, v_{i,\alpha}, v_{j,\beta}\}$ or $\{v_{i,\alpha}, v_{k,\beta}, v_{j,\alpha}\}$. □

Lemma 58. *[22] If H is connected and $|H| \geq 2$, then*

$$\operatorname{aw}(P_2 \square H, 3) \leq 4.$$

Proof. Let c be an exact 4-coloring of $P_2 \square H$ where H_1 and H_2 are the labeled copies of H. If $|c(V(H_1))| \geq 3$, then, by Lemma 56, there exists an isometric C_3 or a shortest isometric path P with at least three colors in H_1. If $C_3 \subseteq H_1$ has three colors, then there is an immediate rainbow 3-AP in $P_2 \square H$. In the other case, $P_2 \square P$ is an isometric subgraph of $P_2 \square H$. By Lemma 49 and Proposition 50 there exists a rainbow 3-AP in $P_2 \square H$ which implies a rainbow 3-AP in $P_2 \square H$ by Proposition 16. In the case where $|c(V(H_1))| = 1$, then $|c(V(H_2))| \geq 3$, which is the previous situation. Finally, consider the case where $|c(V(H_1))| = 2$. Since c is an exact 4-coloring of $P_2 \square H$, $|c(V(H_1)) \backslash c(V(H_2))| = 2$ so by Lemma 40 there is a rainbow 3-AP. □

The results established thus far come together to show an extremely useful bound on the Cartesian products of graphs in Theorem 59. This bound demonstrates that the anti-van der Waerden number of any Cartesian product is either 3 or 4.

Theorem 59. *[22] If G and H are connected graphs and $|G|, |H| \geq 2$, then*

$$\text{aw}(G\square H, 3) \leq 4.$$

Proof. If $|H| = |G| = 2$, then $G\square H = P_2\square P_2$ and By Theorem 13 $\text{aw}(P_2\square P_2, 3) = \text{aw}(C_4, 3) = 3$. Let c be an exact 4-coloring of $G\square H$ with $V(H) = \{v_1, v_2, \ldots, v_n\}$, and suppose without loss of generality $n \geq 3$. If $|G| = 2$, then by Lemma 58 there is a rainbow 3-AP. Now, suppose $|H|, |G| \geq 3$ and define $G_1, G_2, \ldots, G_n$ as the labeled copies of G in $G\square H$. Let $\mathcal{P}$ be the shortest isometric path that contains the most colors in some G_i and consider the following cases.

Case 1: $\mathcal{P}$ has 3 or 4 colors.

Let $\mathcal{P}$ be in G_i and $\mathcal{P}$ has x vertices, and suppose $v_iv_j \in E(H)$. Let $\mathcal{P}'$ be the path in G_j that corresponds to $\mathcal{P}$, note this creates an isometric subgraph $P_2\square P_x$ in $G\square H$. If x is even, then there is a rainbow 3-AP by Proposition 50. If x is odd, then a rainbow 3-AP is guaranteed by Lemma 49 since path $\mathcal{P}$ has 3 or 4 colors.

Case 2: $\mathcal{P}$ is monochromatic.

This implies that each G_i is monochromatic by the definition of $\mathcal{P}$. Since $G\square H$ has 4 colors, there exists either an isometric C_3 or an isometric shortest path $\mathcal{P}'$ in a copy of H that has at least 3-colors by Lemma 56. If there is an isometric C_3, then there is an immediate rainbow 3-AP. In the other case, this is just Case 1 with the roles of G and H reversed.

Case 3: $\mathcal{P}$ has two colors.

In this case, some copy of G has exactly two colors, call this copy G_d and assume the two colors are *red* and *blue*. By Lemma 40, when the remaining two new colors appear they must both appear either with colors *red* or *blue*. Let *purple* and *green* be the two additional colors that are introduced and, without loss of generality, suppose they both appear with *red*. In particular, let $c(V(G_e)) = \{red, green\}$ and $c(V(G_f)) = \{red, purple\}$. Now, create an auxiliary coloring c' of H defined by

$$c'(v_\ell) = \begin{cases} red & \text{if } c(V(G_\ell)) = \{red\} \\ \mathcal{C} & \text{if } c(V(G_\ell)) = \{\mathcal{C}, red\} \end{cases}.$$

Subcase 1: There is no path in H, under coloring c', that contains the colors *blue*, *green*, and *purple*.

Find the smallest subgraph of H that contains *blue*, *green*, and *purple*, say $c'(v_i) = blue$, $c'(v_j) = green$ and $c'(v_k) = purple$ and call this smallest subgraph K. This guarantees that v_i, v_j and v_k are leaves in the subgraph K. Without loss of generality, assume $\mathrm{d}_K(v_i, v_j) \leq \mathrm{d}_K(v_j, v_k)$. Let $v_{i,\alpha} \in G_i$ such that $c(v_{i,\alpha}) = blue$, $v_{j,\beta} \in G_j$ such that $c(v_{j,\beta}) = green$ and $v_{i,\beta}$ be the vertex in G_i that corresponds to $v_{j,\beta}$. Let $v_{k,\alpha}$ be the vertex in G_k that corresponds to $v_{i,\alpha}$ and find a shortest path P from $v_{j,\beta}$ to $v_{k,\alpha}$ whose only vertex in G_j is $v_{j,\beta}$. Now, consider the 3-AP, $\{v_{i,\alpha}, v_{j,\beta}, u\}$, such that u is a vertex on P since $\mathrm{d}(v_i, v_j) \leq \mathrm{d}(v_j, v_k)$. If $c(u) = blue$ or $c(u) = green$ this contradicts the minimality of K or the assumption of the subcase. Therefore, $c(u) \in \{red, purple\}$ and this 3-AP is rainbow.

Subcase 2: There is a path in H, under coloring c', that contains *blue*, *green*, and *purple*.

Let $\mathbb{P}$ be the shortest path in H that contains *blue*, *green* and *purple*. Without loss of generality, assume the path has leaves v_i and v_k with $c'(v_i) = blue$ and $c'(v_k) = purple$. Further, assume v_j is the closest *green* vertex to v_i on $\mathbb{P}$ and $\mathrm{d}(v_i, v_j) \leq \mathrm{d}(v_j, v_k)$. Note, there are no other *blue* or *purple* vertices on $\mathbb{P}$, otherwise $\mathbb{P}$ would not be the shortest path that contains *blue*, *green*, and *purple*. Let $v_{i,\alpha}$ and $v_{j,\beta}$ be in G_i and G_j, respectively, so that they are the closest two vertices with $c(v_{i,\alpha}) = blue$ and $c(v_{j,\beta}) = green$ (see Figure 3.4 for the following construction). Let P be a shortest path from $v_{i,\alpha}$ to $v_{i,\beta}$ in G_i and P' be a shortest path from $v_{i,\beta}$ to $v_{j,\beta}$. Notice that, by minimality of distance from v_i to v_j, $P \square P'$ is an isometric subgraph of $G \square H$. Note that the length of P' is 1 then there is a rainbow 3-AP by Lemma 57. Assume the length of P' is at least 2. If $\mathrm{d}(v_{i,\alpha}, v_{j,\beta})$ is even, then there is a *red* vertex in $P\Box P'$, say u, such that $\mathrm{d}(v_{i,\alpha}, u) = \mathrm{d}(u, v_{j,\beta})$ which creates a rainbow 3-AP.

Now, consider the case where $\mathrm{d}(v_{i,\alpha}, v_{j,\beta}) = 2x + 1$. Let $v_{k,\gamma}$ be a vertex in G_k such that $\mathrm{d}(v_{j,\beta}, v_{k,\gamma})$ is minimal and $c(v_{k,\gamma}) = purple$. Let ρ be a shortest path from $v_{j,\beta}$ to $v_{j,\gamma}$ in G_j and ρ' be a shortest path from $v_{j,\gamma}$ to $v_{k,\gamma}$, then $\rho\Box\rho'$ is an isometric subgraph of $G\Box H$. Note that $c(V(G_{k-1})) = \{red\}$ and $c(V(H_{\gamma-1})) = \{red\}$ by Lemma 57. Define $D_a = \{v \in V(\rho\Box\rho') | \, \mathrm{d}(v, v_{j,\beta}) = a\}$ and note that this means $D_0 = \{v_{j,\beta}\}$. Define y so that $D_y = \{v_{k,\gamma}\}$. Further, define the distance from D_s to D_t to be $|s - t|$. If $y < 2x + 1$, let u be the vertex on P' or P such that $\mathrm{d}(u, v_{j,\beta}) = y$. Then, $c(u) \in \{red, blue\}$ and $\{v_{k,\gamma}, v_{j,\beta}, u\}$ is a rainbow 3-AP. This means $D_{2x+1} \neq \emptyset$, further, $c(D_{2x+1}) = \{green\}$ because if $v \in D_{2x+1}$ such that $c(v) = red$, then $\{v_{i,\alpha}, v_{j,\beta}, v\}$ is a rainbow 3-AP. This implies that the distance from D_y to either D_0 or D_{2x+1} is even. If $y - 0$ is even, then either

$$\{v_{k,\gamma}, v_{k-1,\gamma-(y/2-1)}, v_{j,\beta}\} \text{ or } \{v_{k,\gamma}, v_{k-(y/2-1),\gamma-1}, v_{j,\beta}\}$$

is a rainbow 3-AP since $c(v_{k-1,\gamma-(y/2-1)}) = c(v_{k-(y/2-1),\gamma-1}) = red$. Similarly, if $y - 2x - 1$ is even and $z = \frac{y-2x-1}{2}$, then either

$$\{v_{k,\gamma}, v_{k-1,\gamma-(z-1)}, v_{k-1-z,\gamma-(z-1)}\} \text{ or } \{v_{k,\gamma}, v_{k-(z-1),\gamma-1}, v_{k-(z-1),\gamma-1-z}\}$$

is a rainbow 3-AP.

Therefore, each case yields a rainbow 3-AP so $\mathrm{aw}(G\Box H, 3) \leq 4$. $\Box$

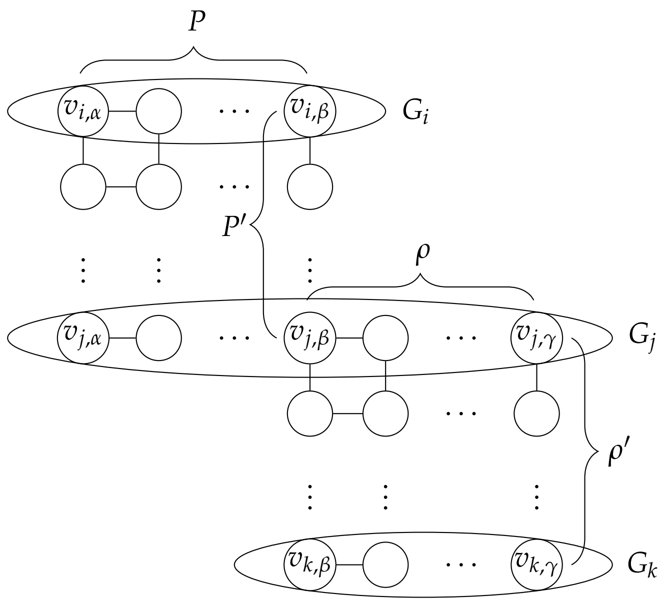

Figure 3.4 Construction of isometric subgraphs of $G\Box H$.

The result of Theorem 59 is used, with earlier results, to determine $\mathrm{aw}(P_m\Box P_n, 3)$ for all m and n. It is interesting to note that the pattern for the small values of m does not continue when considering large values of m. Essentially, there are 'more' 3-APs which forces the anti-van der Waerden number to always be 4. First notice that Lemma 51 and Theorem 59 give the result of Corollary 60 immediately.

Corollary 60. *[22] If $G = P_m\Box P_n$ and $m + n = 2\ell + 1$ for some $\ell \geq 1$, then*

$$\mathrm{aw}(G, 3) = 4.$$

Lemma 61 gives the final lower bound to determine the anti-van der Waerden number for all $P_m\Box P_n$.

Lemma 61. *[22] If $m \geq 4$, $n \geq 4$ and $m + n = 2\ell$ for some $\ell \geq 1$, then*

$$4 \leq \mathrm{aw}(P_m\Box P_n, 3).$$

Proof. Let $G = P_m \square P_n$ and define

$$c(v_{i,j}) = \begin{cases} red & \text{if } i = 1 \text{ and } j = 2 \text{ or } i = 2 \text{ and } j = 1 \\ blue & \text{if } i = m \text{ and } j = n \\ green & \text{otherwise} \end{cases}.$$

Note that if a rainbow 3-AP exists it must contain vertex $v_{m,n}$ and either $v_{1,2}$ or $v_{2,1}$. Let $S = \{v_{m,n}, v_{1,2}, v_{2,1}\}$. Note that $\mathrm{d}(v_{1,2}, v_{m,n}) = \mathrm{d}(v_{2,1}, v_{m,n}) = m + n - 3$ which, by assumption, is odd. Therefore, there does not exist a vertex equidistant from $v_{2,1}$ and $v_{m,n}$ or equidistant from $v_{1,2}$ and $v_{m,n}$. This means a rainbow 3-AP cannot exist in the order of $\{v_{2,1}, v_{i,j}, v_{m,n}\}$ or $\{v_{1,2}, v_{i,j}, v_{m,n}\}$. This means any rainbow 3-AP must exist in the order of $\{v_{m,n}, v_{2,1}, v_{i,j}\}$ or $\{v_{m,n}, v_{1,2}, v_{i,j}\}$ (or the reverse order) where $v_{i,j} \notin S$. Note that $v_{i,j}$ must be distance $m + n - 3$ from one of the vertices in S, but the only vertices distance $m + n - 3$ from any vertex in S are already in S thus $v_{i,j}$ does not exist. Therefore, c avoids rainbow 3-APs so $4 \leq \mathrm{aw}(G, 3)$. $\square$

Using Theorem 59, Corollary 60 and Lemma 61 gives Corollary 62.

Corollary 62. *[22] If $m \geq 4$ and $n \geq 4$, then*

$$\mathrm{aw}(P_m \square P_n, 3) = 4.$$

Combining Propositions 50, 52, 53, 55 and Corollary 62 gives a function to determine $\mathrm{aw}(P_m \square P_n, 3)$ for all m and n.

Theorem 63. *[22] For $2 \leq m \leq n$,*

$$\mathrm{aw}(P_m \square P_n, 3) = \begin{cases} 3 & m = 2 \text{ and } n \text{ is even or } m = 3 \text{ and } n \text{ is odd} \\ 4 & \text{otherwise.} \end{cases}$$

3.5 Graph Families

In this section, the anti-van der Waerden number is determined for complete binary trees and hypercubes. The anti-van der Waerden number for a variety of graph families are then listed in Table 3.5.

Let $\mathcal{B}_n$ be the complete binary tree on $2^{n+1}-1$ vertices. Note $\mathrm{aw}(\mathcal{B}_0,3)=2$ and $\mathrm{aw}(\mathcal{B}_1,3)=3$ by Theorem 12.

Observation 64. *[27]* $\mathrm{aw}(\mathcal{B}_2,3)=4$.

Observation 64 has an upper bound of 4 by Theorem 34 and the lower bound is given in Figure 3.5.

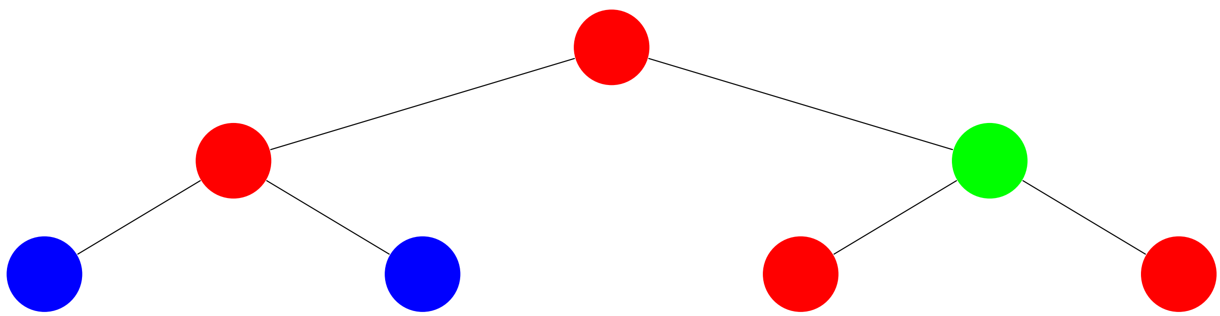

Figure 3.5 A 3-coloring of $\mathcal{B}_2$ with no rainbow 3-AP.

Proposition 65. *[27]* $\mathrm{aw}(\mathcal{B}_n,3)=4$ *for all* $n\geq 2$.

Proof. For the upper bound, $\mathrm{aw}(\mathcal{B}_n,3)\leq 4$ by Theorem 34 since $\mathcal{B}_n$ has no bijacent vertices. The case when $n=2$ is shown in Observation 64. Let $n\geq 3$ and v_0 be the root of $\mathcal{B}_n$, then following cases give an exact 3-coloring with no rainbow 3-APs. If n is even, let

$$c(w) := \begin{cases} \textit{red} & w=v_0 \\ \textit{blue} & \text{if } \mathrm{d}(w,v_0)=n-1 \\ \textit{green} & \text{otherwise.} \end{cases}$$

If n is odd, let

$$c(w) := \begin{cases} \textit{red} & w=v_0 \\ \textit{blue} & \text{if } \mathrm{d}(w,v_0)=n \\ \textit{green} & \text{otherwise.} \end{cases}$$

$\square$

Another family of graphs investigated were *hypercubes*. The hypercube on 2 vertices is denoted Q_1 and $Q_1=K_2$. For larger hypercubes, define $Q_n=Q_{n-1}\square K_2$ where Q_n denotes the

n-dimensional hypercube. Note that Q_n has 2^n vertices. Label the vertices with binary strings of length n such that the distance between two vertices is equal to the number of bits they differ by. Observe that the diameter of Q_n is n.

Let $\vec{0}_n$ and $\vec{1}_n$ be the vertices of Q_n with all 0s and 1s, respectively. For each vertex $v \in V(Q_n)$, define $|v| := \mathrm{d}(v, \vec{0}_n)$. Let $\overline{x} \in V(Q_n)$ be the vertex obtained by switching all n bits of x, i.e. the unique vertex that is distance n from x.

Let $L_i = \{x \in V(Q_n) : |x| = i\}$, i.e. the set of vertices with i 1s.

Proposition 66. *For all $\ell \geq 1$,*

$$4 \leq \mathrm{aw}(Q_{2\ell+1}, 3).$$

Proof. Let $\vec{0}_{2\ell+1} = u$ and $\vec{1}_{2\ell+1} = v$. Note that $\mathrm{d}(u, v) = 2\ell + 1$ and that no other vertex in $Q_{2\ell+1}$ is distance $2\ell + 1$ from u or v. Define the coloring c such that $c(u) = \mathit{red}, c(v) = \mathit{blue}$ and all other vertices get colored *green*. Note that in order for a 3-AP to be rainbow it must contain u and v. Since the distance between u and v is odd u and v cannot be the ends of the 3-AP. However, the only other 3-APs with u and v are $\{u, v, u\}$ and $\{v, u, v\}$ thus c has no rainbow 3-AP. □

Combining Theorem 59 and Proposition 66 yields Corollary 67.

Corollary 67. *For all $\ell \geq 1$,*

$$\mathrm{aw}(Q_{2\ell+1}, 3) = 4.$$

Moreover, the extremal coloring of $Q_{2\ell+1}$ that avoids rainbow 3-APs is unique up to isomorphism.

Proposition 68. *[27] For all $\ell \geq 1$,*

$$\mathrm{aw}(Q_{2\ell}, 3) = 3.$$

Proof. Let S_0 be the subset of vertices in $V(Q_{2\ell})$ with 0 as the first bit, and S_1 be the subset of vertices in $V(Q_{2\ell})$ with 1 as the first bit. Let L_i be the subset of vertices in $V(Q_{2\ell})$ with i bits that are 1s. Suppose c is an exact 3-coloring of $Q_{2\ell}$ with no rainbow 3-APs.

If $|c(S_0)| = 3$, then by Corollary 67, S_0 must have the unique extremal coloring. In particular, $c(0\vec{0}_{2\ell-1}) = \mathit{green}$, $c(0\vec{1}_{2\ell-1}) = \mathit{blue}$, and the remaining vertices of S_0 are colored *red*.

If $c(\vec{1}_{2\ell}) = blue$, then $\{\vec{0}_{2\ell}, \vec{0}_\ell\vec{1}_\ell, \vec{1}_{2\ell}\}$ is a rainbow 3-AP. If $c(\vec{1}_{2\ell}) = green$, then $\{00\vec{1}_{2\ell-2}$,$01\vec{1}_{2\ell-2}$, $11\vec{1}_{2\ell-2}\}$ is a rainbow 3-AP. If $c(\vec{1}_n) = red$ and $x \in L_{2\ell-1}$ such that $c(x) \neq blue$ then, either $\{0\vec{1}_{2\ell-1}, \vec{0}_{2\ell}, x\}$ or $\{0\vec{1}_{2\ell-1}, \vec{1}_{2\ell}, x\}$ is a rainbow 3-AP. Therefore, $c(L_{2\ell-1}) = \{blue\}$ and either $\{000\vec{0}_{2\ell-3}, 110\vec{0}_{2\ell-3}, 011\vec{0}_{2\ell-3}\}$, $\{11\vec{0}_{2\ell-2}, 10\vec{1}_{2\ell-2}, 00\vec{0}_{2\ell-2}\}$, or $\{11000\vec{0}_{2\ell-5}, 11011\vec{1}_{2\ell-5}, 00011\vec{0}_{2\ell-5}\}$ is a rainbow 3-AP when $c(110\vec{0}_{2\ell-3})$ is *blue*, *red*, or *green*, respectively. Thus any 3-coloring of $Q_{2\ell}$ yields a rainbow 3-AP when $\ell \geq 2$.

Therefore, without loss of generality, either $|c(S_0)| = 2$ and $|c(S_1)| = 1$, or $|c(S_0)| = 2$ and $|c(S_1)| = 2$.

If $c(S_0) = \{red, green\}$ and $c(S_1) = \{blue\}$, then without loss of generality assume $c(\vec{0}_{2\ell}) = green$. For some i there exists an element $x \in L_i \cap S_0$ such that $c(x) = red$. Then for any $y \in L_i \cap S_1$, the 3-AP $\{x, \vec{0}_{2\ell}, y\}$ is a rainbow since $c(y) = blue$. This is a contradiction.

Now assume $c(S_0) = \{red, green\}$ and $c(S_1) = \{red, blue\}$. There is a vertex x with $c(x) = green$ and $c(\overline{x}) \neq green$. So there is an automorphism of the vertex set such that $c(\vec{0}_{2\ell}) \neq c(\vec{1}_{2\ell})$.

If $c(\vec{0}_{2\ell}) = green$ and $c(\vec{1}_{2\ell}) = red$, then a layer L_i, with $1 \leq i \leq 2\ell - 1$, that contains an element with the color *blue* must also contain an element with color *red* in $L_i \cap S_0$. This yields a rainbow 3-AP with $\vec{0}_{2\ell}$. A similar argument holds if $c(\vec{0}_{2\ell}) = red$ and $c(\vec{1}_{2\ell}) = blue$.If $c(\vec{0}_{2\ell}) = green$ and $c(\vec{1}_{2\ell}) = blue$, then for each i, L_i is one of the following two types:

$$\begin{aligned} &type\, 1: \quad c(L_i) = \{red\}, \\ &type\, 2: \quad c(L_i \cap S_0) = \{green\} \text{ and } c(L_i \cap S_1) = \{blue\}. \end{aligned}$$

Since $|c(Q_{2\ell})| = 3$, a layer of type 1 must exist. If a layer of type 2 exists, then either there exists an i such that L_i is type 1 and L_{i+1} is type 2, or there exists an i such that L_i is type 2 and L_{i+1} is type 1. In the first case, if $01\vec{x} \in V(L_{i+1})$, then $\{01\vec{x}, 00\vec{x}, 10\vec{x}\}$ is a rainbow 3-AP since $c(01\vec{x}) = green$, $c(00\vec{x}) = red$, and $c(10\vec{x}) = blue$. In the second case, if $11\vec{x} \in V(L_{i+1})$, then $\{01\vec{x}, 11\vec{x}, 10\vec{x}\}$ is a rainbow 3-AP since $c(01\vec{x}) = green$, $c(11\vec{x}) = red$, and $c(10\vec{x}) = blue$. So there does not exist a layer of type 2 and the only vertices not colored *red* are $\vec{0}_{2\ell}$ and $\vec{1}_{2\ell}$.

However, since 2ℓ is even this coloring has a rainbow 3-AP, $\{\vec{0}_{2\ell}$, x, $\vec{1}_{2\ell}\}$, where $x \in L_\ell$.

In each case, c has a rainbow 3-AP, therefor $\text{aw}(Q_{2\ell}, 3) = 3$. □

Table 3.5 lists the known anti-van der Waerden numbers of various graph families.

captionGraph families and their anti-van der Waerden numbers for $k = 3$.

Class of Graphs	anti-van der Waerden number
Graph Join $G + H$	3
Complete Multipartite Graph	3
Radius 1 Graphs	3
Diameter 2 Graphs	3
Diameter 3 Trees	4
Trees with diameter $2\ell + 1$ with no degree 2 vertices	4
Binary Tree $\mathcal{B}_n$, $n \geq 2$	4
Hypercube Q_n	$\begin{cases} 3 & \text{if } n \text{ is even,} \\ 4 & \text{if } n \text{ is odd.} \end{cases}$
Path on n vertices, $7 \cdot 3^{m-2} + 1 \leq n \leq 21 \cdot 3^{m-2}$	$\begin{cases} m+2 & \text{if } n = 3^m \\ m+3 & \text{otherwise} \end{cases}$
Cycle on n vertices	$\mathrm{aw}(\mathbb{Z}_n, 3)$ [7, Corollary 3.15]

CHAPTER 4. THE ANTI-VAN DER WAERDEN NUMBER WITH $k > 3$

Thus far, the anti-van der Waerden number has been primarily found for $k = 3$. In this chapter, the exploration of the anti-van der Waerden number is done for $k \geq 4$ and results and techniques are given.

4.1 The anti-van der Waerden Number of $[n]$ with $k \geq 4$

Table 4.1 gives values of $\mathrm{aw}([n], k)$ for $3 \leq k \leq \frac{n+3}{2}$.

Table 4.1 [26] Values of $\mathrm{aw}([n], k)$ for $3 \leq k \leq \frac{n+3}{2}$.

$n \setminus k$	3	4	5	6	7	8	9	10	11	12	13	14
3	3											
4	4											
5	4	5										
6	4	6										
7	4	6	7									
8	5	6	8									
9	4	7	8	9								
10	5	8	9	10								
11	5	8	9	10	11							
12	5	8	10	11	12							
13	5	8	11	11	12	13						
14	5	8	11	12	13	14						
15	5	9	11	13	14	14	15					
16	5	9	12	13	15	15	16					
17	5	9	13	13	15	16	16	17				
18	5	10	14	14	16	17	17	18				
19	5	10	14	15	17	17	18	18	19			
20	5	10	14	16	17	18	19	19	20			
21	5	11	14	16	17	19	20	20	20	21		
22	6	12	14	17	18	20	21	21	21	22		
23	6	12	14	17	19	20	21	22	22	22	23	
24	6	12	15	18	20	20	22	23	23	23	24	
25	6	12	15	19	21	21	23	23	24	24	24	25

4.2 The anti-van der Waerden Number of $\mathbb{Z}_n$ with $k \geq 4$

In [7], results were found for the anti-van der Waerden number of $\mathbb{Z}_n$ with $k \geq 4$ found computationally. These results appear in Table 4.2.

Table 4.2 [7] Computed values of $\mathrm{aw}(\mathbb{Z}_n, k)$ for $k \geq 4$.

$n \setminus k$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
4	4															
5	4	5														
6	5	5	6													
7	4	5	6	7												
8	6	6	7	7	8											
9	5	6	8	8	8	9										
10	6	8	8	8	9	9	10									
11	5	6	7	8	9	9	10	11								
12	8	9	10	10	11	11	11	11	12							
13	5	7	8	9	10	10	11	11	12	13						
14	6	8	10	12	12	12	12	12	13	13	14					
15	8	11	12	12	12	13	14	14	14	14	14	15				
16	8	10	10	11	14	14	14	14	15	15	15	15	16			
17	6	8	10	11	12	12	13	14	14	15	15	15	16	17		
18	8	10	13	14	14	16	16	16	17	17	17	17	17	17	18	
19	6	9	10	12	12	14	14	15	16	16	16	17	17	17	18	19

4.3 The anti-van der Waerden Number of Graphs with $k \geq 4$

In the investigation of the anti-van der Waerden number of graphs in Chapter 3, the majority of the results focus on k-APs where $k = 3$. In this section, the study of the anti-van der Waerden number of graphs is explored for $k \geq 4$. Determining techniques for finding $\mathrm{aw}(G, k)$ where $k \geq 4$ is a relatively unexplored area. Researching 3-term arithmetic progressions of a graph can be simplified by focusing on the central vertex of the progression, then finding equidistant neighbors such that all 3 vertices have distinct colors. However, in higher term arithmetic progressions this technique can not be used. This section seeks new techniques for these higher term progressions.

In Chapter 3 the only results that were given for k-APs for a general k were Observation 14, Lemma 15, Proposition 16, and Proposition 17. In this section, preliminary results of the study of the anti-van der Waerden number of graphs with $k \geq 4$ are given.

First observe that $\text{aw}(K_n, k) = k$, since any k vertices with distinct colors form a rainbow k-AP (Observation 69). However, even an early investigation of the complete bipartite graph $K_{m,n}$ varies from the change between 3-APs and 4-APs as seen in Proposition 70.

Observation 69. *For $n \geq k \geq 1$,*

$$\text{aw}(K_n, k) = k.$$

This example of a variation leads to a more thorough examination of k-APs with $k \geq 4$.

Proposition 70. *For $m + n \geq 5$,*

$$\text{aw}(K_{m,n}, 4) = 5.$$

Proof. Let X and Y be the independent sets of $K_{n,m}$. Let c be an exact 4-coloring with $c(X) = \{red\}$ and $c(Y) = \{blue, green, purple\}$. Notice for a 4-AP the distance is either 1 or 2 since $\text{diam}(K_{m,n}) = 2$. Let $x_1, x_2, x_3, x_4 \in X$ and $y_1, y_2, y_3, y_4 \in Y$. Any 4-AP of $K_{m,n}$ has the form of either $\{x_1, y_1, x_2, y_2\}$, $\{x_1, x_2, x_3, x_4\}$, or $\{y_1, y_2, y_3, y_4\}$. In the first case both x_1 and x_2 are *red* since $c(X) = \{red\}$. Similarly, in the second case all 4 vertices are *red*. In the final case at least one color is repeated since $|c(Y)| = 3$. Therefore $\text{aw}(K_{m,n}, 4) \geq 5$.

Suppose c is an exact 5-coloring of $K_{m,n}$ such that there are no rainbow 4-APs. Notice if $|c(x)| \geq 4$ or $|c(y)| \geq 4$ there is a rainbow 4-AP, so both $|c(X)| \leq 3$ and $|c(Y)| \leq 3$. Since c is a 5-coloring, at least one of them is equal to 3, so suppose without loss of generality $|c(Y)| = 3$. Again since c is a 5-coloring $|c(X)| \geq 2$. Let $c(x_1) = red$, $c(x_2) = blue$, $c(y_1) = green$, $c(y_2) = purple$, and $c(y_3) = yellow$. There are then a multitude of rainbow 4-APs, and thus c can not be a 5-coloring with no rainbow 4-AP. $\square$

Much like the investigation of 3-term arithmetic progressions, the investigation of k-term arithmetic progressions begins with an examination of graphs with dominating vertices and graphs G such that $\text{diam}(G) = 2$.

4.4 Dominating Vertices and Diameter 2 Graphs

In this section, the anti-van der Waerden number is determined for graphs with a dominating vertex and graphs with diameter 2.

4.4.1 Dominating Vertices

Recall that a dominating vertex v of a graph is a vertex that is adjacent to all other vertices of the graph. Observation 18 leads to Conjecture 71 when considering k-term arithmetic progressions for all $k \geq 3$.

Conjecture 71. *If G is a graph with a dominating vertex, then*

$$k \leq \mathrm{aw}(G,k) \leq k+1.$$

Note that Conjecture 71 is true for $k=3$ by Observation 18 and $k=4$ and 5, shown in Proposition 72.

Proposition 72. *If G is a graph with a dominating vertex and $k \in \{4,5\}$, then*

$$k \leq \mathrm{aw}(G,k) \leq k+1.$$

Proof. Let G be a graph with dominating vertex v. Note that the lower bounds are a consequence of Observation 14.

First consider $k=4$ and let c be an exact 5-coloring of G. Since v is dominating and G has 5 colors, v must have four neighbors whose colors are distinct from each other and distinct from $c(v)$. If those neighbors form an independent set, then they form a rainbow 4-AP. If any pair of them are adjacent, then they form a rainbow 4-AP that includes v. In all cases we have found a rainbow 5-AP thus $\mathrm{aw}(G,4) \leq 5$.

Now, assume $k=5$ and c is an exact 6-coloring of G. Then v must have five neighbors, v_1, v_2, v_3, v_4, and v_5, whose colors are distinct from each other and distinct from $c(v)$. If v_1, v_2, v_3, v_4, and v_5 form an independent set, then $\{v_1,v_2,v_3,v_4,v_5\}$ is a rainbow 5-AP. If exactly one edge exists between the five vertices, say v_1v_2, then $\{v_1, v_5, v_4, v_3, v_2\}$ is a rainbow 5-AP. If there are exactly two edges between the five vertices that share a vertex, say v_1v_2 and v_2v_3, then $\{v_1, v_4, v_2, v_5, v_3\}$ is a rainbow 5-AP. Finally if there are at least two vertex disjoint edges between the five vertices, say v_1v_2 and v_3v_4, then $\{v_1, v_2, v, v_3, v_4\}$ is a rainbow 5-AP. In all cases we have found a rainbow 5-AP thus $\text{aw}(G, 5) \le 6$. □

Lemma 73. *If G has a dominating vertex and $\alpha(G) \ge k \ge 4$, then*

$$\text{aw}(G, k) \ge k + 1.$$

Proof. Let G be a graph with dominating vertex v_0 and $\alpha(G) = \ell \ge k$. Let $v_1, v_2, \ldots, v_\ell$ form an independent set of G. Let c be the following k-coloring of G,

$$c(u) := \begin{cases} i & \text{if } u = v_i \\ k & \text{otherwise.} \end{cases}$$

Any k-AP of G must use vertices $v_0, v_1, \ldots, v_{k-1}$ since these vertices are all uniquely colored. However,, vertices $v_1, v_2, \ldots, v_{k-1}$ are all distance 2 from one another and all distance 1 to vertex v_0. Therefore, there is no k-AP which contains all the vertices $v_0, v_1, \ldots, v_{k-1}$, therefore there can be no rainbow k-AP. □

An examination of graphs with dominating vertices would not be complete without also examining star graphs. In Proposition 74, the anti-van der Waerden number is determined for stars when $k \ge 4$. Note when $k = 3$, the anti-van der Waerden number of a star graph is given by Observation 18.

Proposition 74. *[27] If $4 \le k \le n + 1$, then*

$$\text{aw}(K_{1,n}, k) = k + 1.$$

Proof. Let $G = K_{1,n}$ with $4 \le k \le n + 1$ and ℓ_0 be the central vertex of G. Label the leaves of G as $\{\ell_1, \ell_2, \ldots, \ell_n\}$ and note that $k - 1 \le n$. Define $c : V(G) \to \{0, 1, 2, \ldots, k - 1\}$

$$c(\ell_i) = \begin{cases} i & \text{if } 0 \le i \le k - 2 \\ k - 1 & \text{otherwise} \end{cases}.$$

Then c is an exact k-coloring where every k-AP has at most $k-1$ colors so $k+1 \leq \mathrm{aw}(G,k)$. If $|G| = k$, then Observation 14 gives $\mathrm{aw}(G,k) \leq |G|+1 = k+1$. Now consider $|G| \geq k+1$ which means $k \leq n$. Let c' be any exact $(k+1)$-coloring of G. There must exist k colors that appear on the leaves and those leaves form a rainbow k-AP, therefore $\mathrm{aw}(G,k) \leq k+1$. Therefore, $\mathrm{aw}(G,k) = k+1$. □

4.4.2 Diameter 2 Graphs

In this subsection, the anti-van der Waerden number for graphs with diameter 2 and $k \geq 4$ are examined.

Proposition 75. *If G is a connected graph with* $\mathrm{diam}(G) = 2$*, then*

$$\mathrm{aw}(G,4) \leq 5.$$

Proof. Let c be an exact 5-coloring of G with no rainbow 4-APs. Let $c(v_0) = \mathit{red}$, $c(v_1) = \mathit{blue}$, $c(v_2) = \mathit{green}$, $c(v_3) = \mathit{purple}$, and $c(v_4) = \mathit{yellow}$. Let H be the induced subgraph on the vertices $\{v_0, v_1, v_2, v_3, v_4\}$. Note that if two vertices in a subgraph of G are not adjacent, than the distance between them is 2 since $\mathrm{diam}(G) = 2$. Now, consider the following cases.

Case 1: The number of vertices in the largest connected component of H is 5.

In this case, H is connected. Since H is connected either H has a path on 4 vertices, which is a rainbow 4-AP, or it is the complete bipartite graph $K_{1,4}$ and has a rainbow 4-AP by Proposition 70.

Case 2: The number of vertices in the largest connected component of H is 4.

This implies H has an isolated vertex and a component on the other 4 vertices. Let v_0 be the isolated vertex and v_1, v_2, v_3, v_4 form the other component. If v_1, v_2, v_3, v_4 contain an induced path, then they form a rainbow 4-AP. If v_1, v_2, v_3, v_4 do not contain an induced path, then they form a $K_{1,3}$. Suppose without loss of generality v_1 is adjacent to all three other vertices, then $\{v_0, v_2, v_3, v_4\}$ forms a rainbow 4-AP.

Case 3: the largest connected component of H has at most 3 vertices.

This implies there are at least 2 components. Suppose without loss of generality v_0 and v_1 are in different components and v_2 and v_3 are also in different components. Notice that v_1 could share a component with either v_2 or v_3, but not both. Without loss of generality, suppose v_1 and v_2 are in different components, then $\{v_0, v_1, v_2, v_3\}$ is a rainbow 4-AP.
In each case there exists a rainbow 4-AP, thus $\text{aw}(G, 4) \leq 5$ whenever $\text{diam}(G) = 2$. □

An interesting direction to take this research would be to determine conditions for determining when $\text{aw}(G, 4) = 4$ and when $\text{aw}(G, 4) = 5$ for all graphs G with $\text{diam}(G) = 2$. One possibility for the beginning of this research would be to explore radius conditions for $\text{aw}(G, k)$ for $k \geq 4$.

4.5 Applications to Ramsey Theory

In this section, connections between the anti-van der Waerden number of a graph and the Ramsey number of multiple paths, $R(P_n, P_n)$, are identified.

Let G be an edge colored graph and H be a subgraph of G, H is *edge monochromatic* if every edge of H is the same color. The *Ramsey number*, $R(k, \ell)$, for positive integers k and ℓ, is the smallest n such that for every 2-edge coloring of K_n there exists a monochromatic subgraph isomorphic to K_k in color 1 or a monochromatic subgraph isomorphic to K_ℓ in color 2.

Let $k_1, k_2, \ldots, k_r$ be positive integers. The Ramsey number, $R(k_1, k_2, \ldots, k_r)$, is the smallest n such that every r-edge coloring of K_n contains an edge monochromatic subgraph isomorphic to K_{k_i} in color i for some $i = 1, \ldots, r$.

Let $G_1, G_2, \ldots, G_r$ be graphs. The Ramsey number, $R(G_1, G_2, \ldots, G_r)$ is the smallest n such that every r-edge coloring of K_n contains an edge monochromatic subgraph isomorphic to G_i in color i for some $i = 1, \ldots, r$.

The following theorem was proved in [11] and was later cited in [21].

Theorem 76. *[11] For $n \geq m \geq 2$,*

$$R(P_m, P_n) = n + \lfloor \frac{m}{2} \rfloor - 1.$$

The techniques used in the proof of Proposition 75 identifies a relationship between the Ramsey number of paths and the anti-van der Waerden number. The following result uses this idea alongside Theorem 76.

Theorem 77. *Let G be a graph such that* $\text{diam}(G) = 2$*, then*

$$k \leq \text{aw}(G,k) \leq k + \lfloor \frac{k}{2} \rfloor - 1.$$

Proof. Let $r = k + \lfloor \frac{k}{2} \rfloor - 1$ and let c be an exact r-coloring of G. Suppose $v_1, v_2, \ldots, v_r \in V(G)$ such that $c(v_i) \neq c(v_j)$ for all $i \neq j$.
Let H be the K_r graph on $v_1, v_2, \ldots, v_r$ such that $v_i v_j$ is *blue* if $v_i v_j \in E(G)$ and $v_i v_j$ is *red* if $v_i v_j \notin E(G)$. Notice an edge monochromatic P_k in H represents a rainbow k-AP in G. By Theorem 76, $R(P_k, P_k) = k + \lfloor \frac{k}{2} \rfloor - 1 = r$. Therefore H has an edge monochromatic P_k which is equivalent to saying G has a rainbow k-AP. □

Corollary 78. *The bounds of Theorem 77 are tight.*

Proof. Let G_1 be obtained from taking a K_k and removing exactly one edge. Let c be an exact k-coloring of G_1, then each vertex is a unique color. Clearly, $\text{diam}(G_1) = 2$ and there is a rainbow k-AP since G_1 contains a Hamiltonian path. Thus, there exists a graph G such that $\text{diam}(G) = 2$ and $\text{aw}(G,k) = k$.
Now suppose G_2 is the complete bipartite graph $K_{n,n}$ where $n \geq k$. Let G_2 have bipartite sets X and Y. First notice that the following $k + \lfloor \frac{k}{2} \rfloor - 2$-coloring of G_2 avoids rainbow k-APs.

$$c = \begin{cases} c(X) = \{1, 2, \ldots, \lfloor \frac{k}{2} \rfloor - 1\} \\ c(Y) = \{\lfloor \frac{k}{2} \rfloor, \lfloor \frac{k}{2} \rfloor + 1, \ldots, k + \lfloor \frac{k}{2} \rfloor - 2\} \end{cases}$$

Notice c avoids rainbow k-APs since a rainbow k-AP of $K_{n,n}$ must either use only vertices of X, only vertices of Y, or alternate vertices between X and Y. Neither of the first two can form a rainbow k-AP since neither X nor Y has k colors. The final k-AP can not happen either since X only has $\lfloor \frac{k}{2} \rfloor - 1$ colors and it needs to have at least $\lfloor \frac{k}{2} \rfloor$ colors. Therefore, c is an exact $k + \lfloor \frac{k}{2} \rfloor - 2$-coloring of G_2 which avoids rainbow k-APs.

Now suppose G_2 has $k + \lfloor \frac{k}{2} \rfloor - 1$-colors. By Theorem 77, $\mathrm{aw}(K_{n,n}, k) = k + \lfloor \frac{k}{2} \rfloor - 1$. Thus, there exists a graph G such that $\mathrm{diam}(G) = 2$ and $\mathrm{aw}(G, k) = k + \lfloor \frac{k}{2} \rfloor - 1$.

Since there exists graphs G_1 and G_2 such that $\mathrm{diam}(G_1) = \mathrm{diam}(G_2) = 2$ and $\mathrm{aw}(G_1, k) = k$ and $\mathrm{aw}(G_2, k) = k + \lfloor \frac{k}{2} \rfloor - 1$, the bounds of Theorem 77 are tight. □

In the proof of Corollary 78, it is shown that the complete bipartite graph yields the upper bound for Theorem 77, this result is shown in Corollary 79.

Corollary 79. *For $m + n \geq k$,*

$$\mathrm{aw}(K_{m,n}, k) = k + \lfloor \frac{k}{2} \rfloor - 1.$$

A similar bound can be found for graphs G with $\mathrm{diam}(G) = 3$. The Ramsey number $R(P_n, P_n, P_n)$ was determined in [14] and cited in [21].

Theorem 80. *[14] For large n,*

$$R(P_n, P_n, P_n) = 2n - 2 + (n \mod 2).$$

Theorem 81. *Let G be a graph such that* $\mathrm{diam}(G) = 3$*, then for large enough k*

$$k \leq \mathrm{aw}(G, k) \leq 2k - 2 + (k \mod 2).$$

Proof. Let $r = 2k - 2 + (k \mod 2)$ and let c be an exact r-coloring of G. Suppose $v_1, v_2, \ldots, v_r \in V(G)$ such that $c(v_i) \neq c(v_j)$ for all $i \neq j$.

Let H be the K_r graph on $v_1, v_2, \ldots, v_r$ such that v_iv_j is *blue* in H if $\mathrm{d}_G(v_i, v_j) = 1$, v_iv_j is *red* in H if $\mathrm{d}_G(v_i, v_j) = 2$, and v_iv_j is *green* in H if $\mathrm{d}_G(v_i, v_j) = 3$. Notice an edge monochromatic P_k in H represents a rainbow k-AP in G. By Theorem 80, $R(P_k, P_k, P_k) = 2k - 2 + (k \mod 2) = r$. Therefore H has an edge monochromatic P_k which is equivalent to saying G has a rainbow k-AP. □

Conjecture 82. *The bounds of Theorem 81 are tight.*

Let $R_d(G)$ denote the Ramsey number $R(G, G, \ldots, G)$ where there are d Gs.

The following theorem is a collection of results compiled in [21] and gives the Ramsey number for multiple paths.

Theorem 83. *[21]*

a. $R_k(P_3) = k + 1 + (k \mod 2)$, $R_k(2P_2) = k + 3$ *for all* $k \geq 1$ *[15].*

b. $R_k(P_4) = 2k + c_k$ *for all* k *and some* $0 \leq c_k \leq 2$*. If* k *is not divisible by* 3*, then* $c_k = 3 - k \pmod 3$ *[15]. Wallis [30] showed* $R_6(P_4) = 13$ *which already implied* $R_{3t}(P_4) = 6t + 1$ *for all* $t \geq 2$*. Independently the case* $R_k(P_4)$ *for* $k \neq 3^m$ *was completed by Lindstr om [18]and later Bierbrauer proved* $R_{3^m}(P_4) = 2 \cdot 3^m + 1$ *for all* $m > 1$*. The case for* $m = 1$ *was shown in [15]and is* $R_3(P_4) = 6$*.*

c. $R_k(P_n) \leq (k - c_k)n$ *for some* $c_k > 0$ *for all fixed* $k \geq 2$ *and large* n *[25].*

The following Theorem generalizes the results of Theorems 77 and 81 and the proof uses the same techniques.

Theorem 84. *If* G *is a graph with* $\operatorname{diam}(G) = d$*, then*

$$k \leq \operatorname{aw}(G,k) \leq R_d(P_k).$$

Proof. By Observation 14 $k \leq \operatorname{aw}(G,k)$.

Let G be a graph with $\operatorname{diam}(G) = d$ and suppose $R_d(P_k) = r$. Suppose c is an exact r-coloring of G. Let H be the complete graph K_r formed in the following way. First, let $v_1, v_2, \ldots, v_r \in V(G)$ such that $c(v_i) \neq c(v_j)$ for all $i \neq j$. Let $v_1, v_2, \ldots, v_r$ be the vertices of H. Suppose c' is a coloring of the edges of H. Let $c'(v_iv_j) = \mathrm{d}_G(v_i, v_j)$. Then $|c'| = d$ since $\operatorname{diam}(G) = d$. Any edge monochromatic path of length k in H represents a k-term arithmetic progression in G. Since each vertex of H is a distinct color under coloring c, this edge monochromatic path is equivalent to a rainbow k-AP in G. Note that such an edge monochromatic path exists since $R_d(P_k) = r$. □

Conjecture 85. *The bounds of Theorem 84 are tight.*

Conjecture 86. *Let* $R_d(P_k) = r$ *and suppose the diameter of a graph is* d*, can the Ramsey construction be used to find graphs* G_i *such that* $\operatorname{aw}(G_i, k) = k + i$ *where* $0 \leq i \leq r - k$*.*

Proposition 87. *If $k \geq 4$, then there exists graphs G_i such that* $\text{aw}(G_i, k) = k + i$ *where* $\text{diam}(G_i) = k - 1$ *for* $0 \leq i \leq k - 3$.

Proof. First consider G_0. Since $k = \text{diam}(G_0) + 1$, let $G_0 = P_k$. Since P_k has exactly k vertices $\text{aw}(P_k, k) = k$.

Let G_1 be a 2-blow-up of P_k. That is, if P_k has vertices $v_1, v_2, \ldots, v_k$, then G_1 has independent sets $V_1, V_2, \ldots, V_k$ where $|V_i| = 2$ for each $1 \leq i \leq k$. Let c be an exact k-coloring of G_1 where $|c(V_i)| = 1$ for $1 \leq i < k$. Moreover, $c(V_1) = c(V_2)$ and $c(V_i) \cap c(V_j) = \emptyset$ for all $2 \leq i < j \leq k$. Finally, $|c(V_k)| = 2$, which yields an exact k-coloring of G_1. Notice any k-AP of G_1 either uses each V_i exactly once, or at least one of the V_i's is repeated. If each V_i contributes exactly 1 vertex to the k-AP, then it is not rainbow since $c(V_1) = c(V_2)$. If any V_i besides V_k is repeated, the k-AP is not rainbow. However, if V_k is repeated, then at least one other V_i must be repeated since $k \geq 4$ and V_k is the final independent set in the blow-up. Therefore, this is an exact k-coloring of G_1 with no rainbow k-APs, see Figure 4.1. If G_1 has $k + 1$ colors used on it, then either two V_i's have two colors in them or one V_i has two colors and the remaining each have a unique color. In either case, G_1 has a rainbow k-AP under this coloring. Thus, $\text{aw}(G_1, k) = k + 1$.

Let G_{r-1} be an r-blow-up of the path graph P_k. Let c be an exact $k + r - 2$-coloring of G_{r-1} with no rainbow k-AP, for $r \geq 2$. The coloring is constructed in the following way. $c(V_1) = c(V_2)$, $c(V_i) \cap c(V_j) = \emptyset$ for $2 \leq i < j \leq k$, and $|c(V_k)| = r$. However, forcing another color in yields a k-AP for each $r + k - 1$-coloring of G_{r-1}.

This method works until $r = k$, since $r \geq k$ implies that a single V_k can yield a rainbow k-AP by using the vertices of V_k. Therefore, if $r = k - 1$ this yields a $k + k - 1 - 2 = 2k - 3$-coloring of G_{k-2} with no rainbow k-AP. $\square$

In Figure 4.1, the label of the vertex denotes the color used on the vertex. For simplicity, numbers will be used to represent colors.

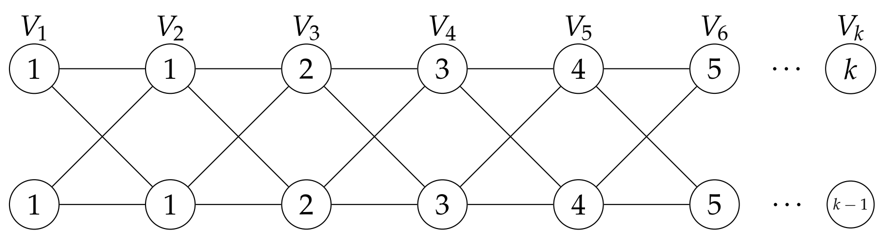

Figure 4.1 A k-coloring of a 2-blow-up of P_k with no rainbow k-AP.

CHAPTER 5. RELATED PROJECTS

Though there has been a lot of research done on the anti-van der Waerden number in the past few years, there are many unexplored areas of research. In this chapter, a few areas of research that the author began, but did not have sufficient time to explore are mentioned with their results.

5.1 The anti-van der Waerden Number of $[m] \times [n]$

An *arithmetic progression* of $[m] \times [n]$ has an initial condition, (a, a'), and a difference term, (d, d'). A $k-$term arithmetic progression, often referred to as a k-AP, is k terms $\{(a, a'), (a + d, a' + d'), \ldots, (a + (k-1)d, a' + (k-1)d')\}$.

An *exact r-coloring* of $[m] \times [n]$ is a surjective function $c : [m] \times [n] \to \{1, 2, \ldots, r\}$. A set of tuples $S \subseteq [m] \times [n]$ is *rainbow* under coloring c, if for any $(s, s'), (t, t') \in S$, $c((s, s')) \neq c((t, t'))$ when $(s, s') \neq (t, t')$.

The *anti-van der Waerden number* of $[m] \times [n]$, denoted by $\text{aw}([m] \times [n], k)$, is the least positive integer r such that every exact r-coloring of $[m] \times [n]$ contains a rainbow k-AP. If $[m] \times [n]$ has no coloring with k-APs, then $\text{aw}([m] \times [n], k) = m \cdot n + 1$.

Note that $[m] \times [n]$ will often be thought of as a grid with m rows and n columns where R_i denotes the ith row.

Proposition 88. *If $m, n \geq k$, then*

$$\text{aw}([m] \times [n], k) \leq \min\{m \cdot \text{aw}([n], k), n \cdot \text{aw}([m], k)\}.$$

Proof. Suppose without loss of generality $n \cdot \text{aw}([m], k) \geq m \cdot \text{aw}([n], k)$. If c is an exact $m \cdot \text{aw}([n], k)$-coloring of $[m] \times [n]$, then by the pigeon hole principle at least one column has at least $\text{aw}([n], k)$ colors used on it and by definition that row contains a rainbow k-AP. $\square$

Proposition 89. *For $n \geq 2$,*

$$\operatorname{aw}([2] \times [n], 3) = 2\operatorname{aw}([n], 3) - 1.$$

Proof. Let c be the following exact $2\operatorname{aw}([n],3) - 2$-coloring of $[2] \times [n]$. First, color row 2 with the extremal $\operatorname{aw}([n],3) - 1$ coloring which avoids rainbow 3-APs. Next, color row 1 with the exact same pattern as row 2, but with $\operatorname{aw}([n],3) - 1$ new colors. This yields an exact $2\operatorname{aw}([n],3) - 2$-coloring of $[2] \times [n]$ which avoids rainbow 3-APs.

Let c be an exact $2\operatorname{aw}([n],3) - 1$-coloring of $[2] \times [n]$. Note by the pigeon hole principle either row 1 or row 2 has at least $\operatorname{aw}([n],3)$ colors used on it and thus contains a rainbow 3-AP. □

The proof of Proposition 89 can be generalized to higher term arithmetic progressions. This is shown in Proposition 90.

Proposition 90. *If $m < k \leq n$, then*

$$\operatorname{aw}([m] \times [n], k) = m \cdot \operatorname{aw}([n], k) - m + 1.$$

Proof. Let c be the following exact $m \cdot \operatorname{aw}([n],k) - m$-coloring of $[m] \times [n]$. Color each row with the extremal $\operatorname{aw}([n],k) - 1$ coloring which avoids rainbow k-APs such that each row is uniquely colored. This yields an exact $m \cdot \operatorname{aw}([n],k) - m$-coloring of $[m] \times [n]$ which avoids rainbow k-APs since no k-AP can use multiple rows because $m < k$.

Let c be an exact $m \cdot \operatorname{aw}([n],k) - m + 1$-coloring of $[m] \times [n]$. Note by the pigeon hole principle at least one row must contain at least $\operatorname{aw}([n],k)$ colors and thus contains a rainbow k-AP. □

Proposition 91. *For $n \geq 3$,*

$$\operatorname{aw}([3] \times [n], 3) \geq \operatorname{aw}([n], 3) + 2.$$

Proof. Let c be the following exact $\operatorname{aw}([n],3) + 1$-coloring of $[3] \times [n]$. First, color row 3 with the extremal $\operatorname{aw}([n],3) - 1$-coloring which avoids rainbow 3-APs. Note that this coloring contains two unitary elements, denote these unitary elements $(3, x)$ and $(3, y)$. Notice that the parity of x and y must be different, otherwise $\{(3, x), (3, \frac{y-x}{2}), (3, y)\}$ is a rainbow 3-AP. Color elements $(1, x)$ and $(2, x)$, a new color such that $c((1, x)) = c((2, x))$ and similarly color $(1, y)$ and $(2, y)$ a new color such that $c((1,y)) = c((2,y))$. For the remaining elements, let $c((i,j)) = c((3,j))$ where $i \in \{1,2\}$, and $j \in [n] \setminus \{x,y\}$.

Suppose for the sake of contradiction c contains the rainbow 3-AP $\{(u,u'),(v,v'),(w,w')\}$. Note that since all three rows have the extremal $\mathrm{aw}([n],3)-1$-coloring on them $u \neq v \neq w$ and $u \neq w$. Therefore suppose without loss of generality $u = 1, v = 2, w = 3$. Since this 3-AP is rainbow each element has a unique color, say $c((1,u')) = red$, $c((2,v')) = blue$ and $c((3,w')) = green$. However, this implies $\{(3,u'),(3,v'),(3,w')\}$ is a rainbow 3-AP since the colors of $(1,u')$ and $(2,v')$ are determined by the corresponding elements in row 3. A contradiction since row 3 was constructed to have no rainbow 3-APs. □

Let R_i denote the ith row and C_j denote the jth column in $[m] \times [n]$.

Lemma 92. *If c is an exact $\mathrm{aw}([n],3)+\ell$-coloring of $[3] \times [n]$ that avoids rainbow 3-APs and all the colors that do not appear in row 2 appear in row 1, then $c(R_3) \subseteq c(R_2)$.*

Proof. Let $x_0, x_1, \ldots, x_\ell$ be the $\ell+1$ colors in row 1 that do not appear in row 2. For an element $(3,i)$ of row 3, form the 3-AP from $(3,i)$ to one of the x_i's. Note that at least one such 3-AP must exist. The middle term is an element of row 2 and since there are no rainbow 3-APs and row 2 can not use color x_i, it must be the case that the color of the row 2 element is the same as the row 3 element. This can be done for each color on row 3. □

The following Conjecture, if true, leads to the proof of Conjecture 94.

Conjecture 93. *If c is a coloring of $[m] \times [n]$ that avoids rainbow 3-APs, then any row with $\mathrm{aw}([n],3)-1$ colors used on it is a unitary coloring.*

Conjecture 94. *For all $n \geq 3$,*

$$\mathrm{aw}([3] \times [n], 3) = \mathrm{aw}([n],3) + 2.$$

Proof that Conjecture 93 implies Conjecture 94. By Proposition 91 $\mathrm{aw}([3] \times [n], 3) \geq \mathrm{aw}(n,3) + 2$.

Let c be an exact $\mathrm{aw}([n],3)+2$-coloring of $[3] \times [n]$ that avoids rainbow 3-APs. First note that each row has at most $\mathrm{aw}([n],3)-1$ colors, otherwise $[3] \times [n]$ would contain a rainbow 3-AP.

Therefore, rows 1 and 3 contain at least 3 colors that do not appear in row 2. Suppose that at least 2 appear in row 1 and at least 1 appears in row 3. Suppose without loss of generality $x \leq y \leq z$ and $c((1,x)) = \mathit{red}$, $c((1,y)) = \mathit{blue}$, and $c((3,z)) = \mathit{green}$. If either x and z or y and z have the same parity, then one of $\{(1,x),(2,\frac{z-x}{2}),(3,z)\}$ or $\{(1,y),(2,\frac{z-y}{2}),(3,z)\}$ is a rainbow 3-AP. If x and y share a parity and z has the other parity, then there exists w such that $x < w < y$ and $c((1,w)) \in \{\mathit{red},\mathit{blue}\}$ and the parity of w is the same as the parity of z. This yields the rainbow 3-AP $\{(1,w),(2,\frac{z-w}{2}),(3,z)\}$. Therefore the colors that appear in rows one and three that do not appear in row two must all appear in row one or in row three.

Suppose without loss of generality that all the colors that do not appear on row 2 appear on row 1. By Lemma 92 notice that the color set of row 3 is a subset of the color set of row 2. Let $(1,x_1),(1,x_2)$, and $(1,x_3)$ be elements of row 1 with the 3 colors that do not appear on row 2. Note that at least two of x_1, x_2, and x_3 share a parity, so there exists an x_4 such that $(1,x_4)$ has a color used on it that is not used on row 2. Notice that the minimum number of elements with the three colors that do not appear on row 2 that appear on row 1 is 4, and these four elements have the following properties. First, two of x_1, x_2, x_3, and x_4 have odd parity and two have even parity. Second, the order of the pairities is either odd, even, odd, even, or it is even, odd, even, odd. Suppose without loss of generality $x_1 < x_2 < x_3 < x_4$, x_1 and x_3 have odd parity, and x_2 and x_4 have even parity.

By Conjecture 93, row two has a unitary coloring. Let $(2,y)$ be the element in row two with a unique color. Suppose without loss of generality that y is odd, then both $\{(1,x_1),(2,y),(3,|2y-x_1|)\}$ and $\{(1,x_3),(2,y),(3,|2y-x_3|)\}$ are 3-APs. However, then either $\{(1,x_1),(2,|x_1-\frac{-2y+x_3+x_1}{2}|),(3,|2y-x_3|)\}$ or $\{(1,x_3),(2,|x_3-\frac{-2y+x_1+x_3}{2}|),(3,|2y-x_1|)\}$ is a rainbow 3-AP since row three does not contain $c((1,x_1))$ or $c((1,x_3))$ and $(2,y)$ is the only element of row two with color $c((2,y))$.

$\square$

5.2 Other Types of Graphs

In this section, the anti-van der Waerden number of other types of graphs will be explored, in particular, weighted and directed graphs. The relationship between the anti-van der Waerden number of a weighted or directed graph and the connection to the anti-van der Waerden number of that graphs underlying graph will be of particular interest. The investigation of the anti-van der Waerden number of hypergraphs would also be an interesting avenue to explore, but the author did not have sufficient time to do so.

5.2.1 Directed Graphs

A *directed graph* is a graph for which each edge is an ordered pair $(u, v) \in V^2$. The *anti-van der Waerden number of a directed graph D*, denoted by $\mathrm{aw}(D, k)$, is the least positive integer r such that every exact r-coloring of D contains a rainbow k-AP. If D has n vertices and no coloring of D contains non-degenerate k-APs, then $\mathrm{aw}(D, k) = n + 1$.

Observation 95. *For a directed graph D and the undirected underlying graph G,*

$$\mathrm{aw}(G, k) \leq \mathrm{aw}(D, k).$$

Notice that any rainbow k-term arithmetic progression of D is a rainbow k-term arithmetic progression of G. Therefore, if every $\mathrm{aw}(D, k)$-coloring of D has a rainbow k-AP, then any $\mathrm{aw}(D, k)$-coloring of G will have a rainbow k-AP.

A related case of this can be found in Section 5.1 where the anti-van der Waerden number of $m \times n$ is explored. However, these are not equivalent. In the case of $[m] \times [n]$ the k-APs are fixed, i.e. if the progression goes to the right two and up one, it must repeat this pattern for the remainder of the progression. However, in a directed graph of $P_m \square P_n$ where all edges are directed up and right, the first part of the progression could go over two and up one, then the next could go up two and over one. This gives more potential k-APs which lowers the anti-van der Waerden number, (Observation 96).

Observation 96. *For $m, n \geq 1$ and directed paths P_m and P_n,*

$$\text{aw}(P_m \square P_n, k) \leq \text{aw}([m] \times [n], k).$$

One interesting direction to take research on directed graphs would be the investigation of the relationship between a grid graph of undirected paths P_m and P_n, directed paths D_m and D_n, and a grid $[m] \times [n]$. As seen in Observations 95 and 96, it is clear

$$\text{aw}(P_m \square P_n, k) \leq \text{aw}(D_m \square D_n, k) \leq \text{aw}([m] \times [n], k),$$

but the question remains as to how close these values are and if they are related.

Another interesting direction research on directed graphs could be taken would be to investigate if there is a relationship between the anti-van der Waerden number of a directed graph and the anti-van der Waerden number of the largest directed path of D (Conjecture 97).

Conjecture 97. *If D is a directed graph and d is the length of the longest directed path of D,*

$$\text{aw}(D, k) \leq \text{aw}([d], k).$$

5.2.2 Weighted Graphs

A *weighted graph* is a graph with a function $w : E(G) \to \mathbb{Z}$, where $w(e)$ is the weight of edge e.

Let W be a weighted graph, then the *distance* between two vertices of W, $\text{d}_W(u, v)$, is the minimum weight sum of all paths between u and v. A *k-term arithmetic progression of a weighted graph G*, k-AP, is a subset of k vertices of G of the form $\{v_1, v_2, \ldots, v_k\}$, where $\text{d}(v_i, v_{i+1}) = d$ for all $1 \leq i < k$.

The *anti-van der Waerden number of a weighted graph W*, denoted by $\text{aw}(W, k)$, is the least positive integer r such that every exact r-coloring of W contains a rainbow k-AP. If W has n vertices and no coloring of W contains non-degenerate k-APs, then $\text{aw}(W, k) = n + 1$.

There is no clear relationship between W and the underlying graph G since the weight of each edge changes the arithmetic progressions of W.

One first attempt at relating the weighted graph to it's underlying graph was to replace the weighted edges with vertices that were degree two. However, this quickly becomes problematic, especially in trees, by Theorem 34.

In fact, there are examples of weighted graphs W and underlying graphs G where $\text{aw}(W,k) \leq \text{aw}(G,k)$ and $\text{aw}(G,k) \leq \text{aw}(W,k)$. For example Observations 98 and 99. Therefore, studying the anti-van der Waerden number of a weighted graph to find bounds on the anti-van der Waerden number of the underlying graph is not possible. However, studying the anti-van der Waerden number of weighted graphs may be a worthwhile study in its own right.

Observation 98. *Let W be a P_9 with weighted edges* $1, 1, 1, 1, 1, 1, 1,$ *and* 8, *and* $G = P_9$. *By Theorem 12* $\text{aw}(G,3) = 4$. *However, any* 3*-AP involving the leaf is degenerate since it must repeat a vertex. Therefore,* $\text{aw}(W,3)$ *is equivalent to* $\text{aw}(P_8,3) = 5$.

Observation 99. *Let W be a P_{10} with weighted edges* $1, 1, 1, 1, 1, 1, 1, 1,$ *and* 9, *and* $G = P_{10}$. *By Theorem 12* $\text{aw}(G,3) = 5$. *However, any* 3*-AP involving the leaf is degenerate since it must repeat a vertex. Therefore,* $\text{aw}(W,3)$ *is equivalent to* $\text{aw}(P_9,3) = 4$.

Besides the example in Observation 98, there are also many example of weighted graphs W with no k-APs, so $\text{aw}(W,k) = |W| + 1$, however; the underlying graphs have k-APs, so $\text{aw}(G,k) \leq k$.

5.3 Rainbow Numbers and anti-Schur Numbers

In this section the concept of the rainbow number and anti-Schur number is discussed. In the consideration of a 3-AP $\{a, a+d, a+2d\}$ it can also be thought of as the equation $a + a + 2d = 2(a+d)$. That is, the 3-AP $\{x_1, x_2, x_3\}$ is equivalent to saying $x_1 + x_3 = 2x_2$. This equation can be generalized for any k, giving the equation $x_1 + x_3 = kx_2$ where x_1, x_2, and x_3, are elements of the mathematical object that is being worked over. The concept of the rainbow number has been studied on $\mathbb{Z}_n$ in [5] and over $[n]$ in [23]. The anti-Schur number of $[n]$ has been studied for $k = 1$ in [9].

For a fixed integer k, a *triple* (x_1, x_2, x_3) is any three elements in G which are a solution to $x_1 + x_2 = kx_3$. When $k = 1$, these are referred to as *Schur triples*. A triple is called a *rainbow triple* under a coloring c when $c(x_1) \neq c(x_2)$, $c(x_1) \neq c(x_3)$, and $c(x_2) \neq c(x_3)$. A coloring is called *rainbow-free* when there does not exist a rainbow triple in G under c. For positive integer k, the *rainbow number* of a mathematical object G, denoted $\text{rb}(G, k)$, is the least positive integer r such that every exact r-coloring of G contains a rainbow triple (x_1, x_2, x_3) such that $x_1 + x_2 = kx_3$ and $x_1, x_2, x_3 \in G$. If such an integer does not exist, $\text{rb}(G, k) = |G| + 1$. A *maximum* coloring is a rainbow-free r-coloring of G where $r = \text{rb}(G, k) - 1$.

For positive integers n and k, the *anti-Schur number*, denoted $\text{as}(G, k)$, is the least positive integer r such that every exact r-coloring of the mathematical object G contains a rainbow set $\{x_1, x_2, \ldots, x_k\}$ such that $\sum_{i=1}^{k-1} x_i = x_k$ and $x_j \in G$ for $1 \leq j \leq k$. If such an integer does not exist, $\text{as}(G, k) = |G| + 1$.

Observation 100. *For a mathematical object G, the rainbow number* $\text{rb}(G, 1)$ *is equivalent to the anti-Schur number* $\text{as}(G, 3)$.

A similar connection exists between the rainbow number and the anti-van der Waerden number.

Observation 101. *For a mathematical object G, the rainbow number* $\text{rb}(G, 2)$ *is equivalent to the anti-van der Waerden number* $\text{aw}(G, 3)$.

Observations 100 and 101 indicate connections between rainbow numbers, anti-Schur numbers, and anti-van der Waerden numbers. However, there are still many unexplored equations such as $\sum_{i=1}^{k-1} x_i = \ell x_k$ for positive integers k and ℓ. The exploration of the anti-van der Waerden type problem on these expressions would be an interesting question for future research.

5.3.1 Rainbow Numbers of $\mathbb{Z}_n$

In this subsection, the primary results of [5] on $\text{rb}(\mathbb{Z}_n, k)$ are given. Bevilacqua, King, Kritschgau, Tait, Tebon, and Young began by finding results on $\text{rb}(\mathbb{Z}_n, 1)$.

Observation 102. *[5] It can be deduced through inspection that* $\mathrm{rb}(\mathbb{Z}_2, 1) = \mathrm{rb}(\mathbb{Z}_3, 1) = 3$.

Theorem 103. *[5] For a prime* $p \geq 5$,

$$\mathrm{rb}(\mathbb{Z}_p, 1) = 4.$$

Combining Observation 102 and Theorem 103 gives the rainbow number for $k = 1$ for all primes. Bevilacqua et al. then used this value to determine the rainbow number $\mathrm{rb}(\mathbb{Z}_n, 1)$ for all n.

Theorem 104. *[5] For a positive integer n with prime decomposition* $n = 2^{e_0} p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s}$,

$$\mathrm{rb}(\mathbb{Z}_n, 1) = 2 + e_0 + \sum_{i=1}^{s} \Big(e_i (\mathrm{rb}(\mathbb{Z}_{p_i}, 1) - 2) \Big).$$

After determining $\mathrm{rb}(\mathbb{Z}_n, 1)$, Bevilacqua et al. proceeded to investigate $\mathrm{rb}(\mathbb{Z}_n, p)$ for prime numbers p. Notice if $p = 2$, this is the anti-van der Waerden number $\mathrm{aw}(\mathbb{Z}_n, 3)$ discussed in Chapter 2.

Theorem 105. *[5] Let p and q be distinct primes, then* $\mathrm{rb}(\mathbb{Z}_q, p) = 4$ *if and only if p and q do not satisfy either of the following conditions:*

- *p generates $\mathbb{Z}_q^*$,*
- *$|p| = \frac{q-1}{2}$ in $\mathbb{Z}_q^*$ and $\frac{q-1}{2}$ is odd.*

Otherwise, $\mathrm{rb}(\mathbb{Z}_q, p) = 3$.

Theorem 106. *[5] For a prime* $p \geq 3$ *and* $\alpha \geq 1$,

$$\mathrm{rb}(\mathbb{Z}_{p^\alpha}, p) = \begin{cases} 3 & p = 3, \alpha = 1 \\ 4 & p = 3, \alpha \geq 2 \\ \frac{p+1}{2} + 1 & p \geq 5 \end{cases}$$

Theorems 105 and 106 combined with Theorem 107 determine exact values for $\mathrm{rb}(\mathbb{Z}_n, p)$ for all positive integers n and primes p.

Theorem 107. *[5] Let p be prime and n be a positive integer, such that n has prime decomposition $n = p^{e_0} \cdot q_1^{e_1} \cdots q_s^{e_s}$, then*

$$\mathrm{rb}(\mathbb{Z}_n, p) = \mathrm{rb}(\mathbb{Z}_{p^{e_0}}, p) + \sum_{i=1}^{s} \Big(e_i(\mathrm{rb}(\mathbb{Z}_{q_i}, p) - 2) \Big).$$

If $e_0 = 0$, let $\mathrm{rb}(\mathbb{Z}_{p^{e_0}}, p) = 2$.

5.3.2 Rainbow Numbers of $[n]$

In this subsection, results on $\mathrm{rb}([n], 3)$ are investigated. These results are from [23].

Observation 108. *[23] If k is odd, triples (x_1, x_2, x_3) such that $x_1 + x_2 = k \cdot x_3$ can only contain three even numbers, or two odd numbers and one even.*

Proposition 109. *For $n \geq 5$,*

$$\mathrm{rb}([n], 3) \geq \lfloor \log_2(n) \rfloor + 3.$$

Proof. First note that if $n \leq 4$, there are no triples so every coloring is rainbow free. Therefore, $\mathrm{rb}([1], 3) = 2$, $\mathrm{rb}([2], 3) = 3$, $\mathrm{rb}([3], 3) = 4$, and $\mathrm{rb}([4], 3) = 5$.
Let $n = 5$, then the only triples of $[n]$ are $(1, 5, 2)$ and $(4, 5, 3)$ since $1 + 5 = 3 \cdot 2$ and $4 + 5 = 3 \cdot 3$. Therefore any rainbow free coloring of [5] both $(1, 5, 2)$ and $(4, 5, 3)$ have at most two colors. Since they overlap with 5, one rainbow free coloring is $c(1) = \mathit{red}$, $c(2) = \mathit{blue}$, $c(3) = \mathit{red}$, $c(4) = \mathit{green}$, and $c(5) = \mathit{red}$. Thus, $\mathrm{rb}([5], 3) \geq 4$. However, any 4-coloring is not rainbow free since either $(1, 5, 2)$ or $(4, 5, 3)$ must have 3-colors by the pigeon hole principle. Therefore, $\mathrm{rb}([5], 3) \leq 4$ and hence $\mathrm{rb}([5], 3) = 4$. Now assume by mathematical induction $\mathrm{rb}([\ell], 3) \geq \lfloor \log_2(\ell) \rfloor + 3$ for all $\ell < n$.
Let c' be a coloring of $\lfloor [\frac{n}{2}] \rfloor$ that is rainbow free and suppose without loss of generality *red* is not used in c'. Consider the following coloring c of $[n]$,

$$c(i) := \begin{cases} \mathit{red} & \text{if } i \text{ is odd} \\ c'(\frac{i}{2}) & \text{if } i \text{ is even.} \end{cases}$$

Note by Observation 108, every triple of $[n]$ uses either two odds or all evens, thus every triple either has two red elements or is a triple of c'. In either case, the triple is not rainbow since c' is rainbow free.

Therefore $\text{rb}([n],3) \geq \text{rb}(\lfloor [\frac{n}{2}] \rfloor, 3) + 1$.

If this process is repeated recursively then it will be repeated $\lfloor \log_2(n) \rfloor$ times, adding 1 each time. The final coloring will be of [2], and since $\text{rb}([2],3) = 3$, this yields

$$\text{rb}([n],3) \geq \lfloor \log_2(n) \rfloor + \text{rb}([2],3) = \lfloor \log_2(n) \rfloor + 3.$$

□

During the exploration of the rainbow number of $[n]$, Riley, Schulte, Weida, and Young noticed that if colors were to appear on the odd integers, that did not appear on the even integers they must appear on either integers that are $1 \pmod 6$ or integers that are $5 \pmod 6$, but not both. If two colors *red* and *blue* appear on a which is $1 \pmod 6$ and b which is $5 \pmod 6$, then $\frac{a+b}{3} \in [n]$ and $\frac{a+b}{3}$ is even, thus $(a, b, \frac{a+b}{3})$ is a rainbow triple. Riley et al. also noticed that when new colors appear on the odd integers, they appear to be on integers greater than $\lfloor \frac{2n}{3} \rfloor$ to optimize the coloring. Lemma 110 demonstrates the latter part of these findings.

Lemma 110. *Let c be a rainbow free coloring of $[n]$ and $c(a)$ and $c(b)$ have colors that are not used on the even integers. If a and b are $1 \pmod 6$ such that $a, b > \frac{n}{2}$, then $b - a \leq \frac{n}{3}$.*

Proof. Let $a = pn$ and $b = qn$, where $\frac{1}{2} \leq p < q \leq 1$, then $q - p \leq 1 - p$. If $\frac{2}{3} \leq p$, then $q - p \leq \frac{1}{3}$. Since $n < 3a - b$, $1 < 3p - q$; so, $q - p < 2p - 1$. Therefore, if $p \leq \frac{2}{3}$, then $q - p \leq \frac{1}{3}$. Therefore, $b - a \leq \frac{n}{3}$. □

Proposition 111. *For $n \geq 5$,*

$$\text{rb}([n],3) \leq \log_2(n) + \lfloor \frac{n}{18} \rfloor + 4.$$

Proof. First note that if $n \leq 4$, there are no triples so every coloring is rainbow free. Therefore, $\text{rb}([1],3) = 2$, $\text{rb}([2],3) = 3$, $\text{rb}([3],3) = 4$, and $\text{rb}([4],3) = 5$.

Let $n = 5$, then the only triples of $[n]$ are $(1,5,2)$ and $(4,5,3)$ since $1 + 5 = 3 \cdot 2$ and $4 + 5 = 3 \cdot 3$. Therefore any rainbow free coloring of [5] both $(1,5,2)$ and $(4,5,3)$ have at most two colors. Since they overlap with 5, one rainbow free coloring is $c(1) = red$, $c(2) = blue$, $c(3) = red$, $c(4) = green$, and $c(5) = red$. Thus, $\text{rb}([5],3) \geq 4$. However, any 4-coloring is not rainbow free since either $(1,5,2)$ or $(4,5,3)$ must have 3-colors by the pigeon hole principle. Therefore, $\text{rb}([5],3) \leq 4$ and hence $\text{rb}([5],3) = 4$. Now assume by mathematical induction $\text{rb}([\ell],3) \leq \lfloor \log_2(\ell) \rfloor + \lfloor \frac{\ell}{18} \rfloor + 4$ for all $\ell < n$.

Assume for the sake of contradiction that c is a rainbow free $\text{rb}(\lfloor [\frac{n}{2}] \rfloor, 3) + \lfloor \frac{n}{18} \rfloor + 1$-coloring of $[n]$. By Lemma 110, any new colors that are introduced on the odd integers must be on odd integers greater than $\frac{2n}{3}$. They also must be on only odds that are $1 \pmod 6$. Let the even integers be colored with c', a rainbow free $\text{rb}(\lfloor [\frac{n}{2}] \rfloor, 3) - 1$-coloring. Color all the odd integers less than $\frac{2n}{3}$ with *red*. Color each odd that is $1 \pmod 6$ and greater than or equal to $\frac{2n}{3}$ with a new unique color. This adds at most $\frac{n}{18}$ new colors since there are at most $\frac{n}{18}$ integers that are $1 \pmod 6$ and greater than or equal to $\frac{2n}{3}$. At this point c contains $\text{rb}(\lfloor [\frac{n}{2}] \rfloor, 3) - 1$ colors on the even integers and $1 + \lfloor \frac{n}{18} \rfloor$ colors on the odd integers. However, c has one more color. Therefore, either the evens have $\text{rb}(\lfloor [\frac{n}{2}] \rfloor, 3)$ colors, a contradiction to them being rainbow free. Otherwise, there exists an integer a such that $a = 3 \pmod 6$ or $a = 5 \pmod 6$ and a has a new color.

If $a = 3 \pmod 6$, then there exists integers b and b' such that $b + b' = a$ and $b, b' \leq n$. Moreover, by Observation 108 b is odd and b' is even. Since b is odd and $b \leq \frac{2n}{3}$, $c(b) = red$. Since b' is even it is a color from c' and is a different color from all the odds, therefore (b, b', a) is a rainbow triple.

If $a = 5 \pmod 6$, then there exists an odd b such that $b = 1 \pmod 6$ and $b \geq \frac{2n}{3}$, therefore b is a unique color. However, $\frac{a+b}{3} \leq n$ and is even, therefore $(a, b, \frac{a+b}{3})$ is a rainbow triple.

In any case there exists a rainbow triple and therefore $\text{rb}([n],3) \leq \text{rb}(\lfloor [\frac{n}{2}] \rfloor, 3) + \lfloor \frac{n}{18} \rfloor + 1$.

If this is done recursively, then

$$\text{rb}([n],3) \leq \lfloor \log_2(n) \rfloor + \text{rb}([2],3) + \lfloor \frac{n}{18} \rfloor + 1 \leq \lfloor \log_2(n) \rfloor + \lfloor \frac{n}{18} \rfloor + 4.$$

□

From Propositions 109 and 111, the rainbow number of $[n]$ is bounded,

$$\lfloor \log_2(n) \rfloor + 3 \leq \text{rb}([n],3) \leq \lfloor \log_2(n) \rfloor + \lfloor \frac{n}{18} \rfloor + 4.$$

Riley et al. hope to extend this work to either tighten these bounds or determine the exact value of $\mathrm{rb}([n], 3)$. If they can accomplish this, they hope to generalize their results to $\mathrm{rb}([n], k)$. Table 5.1 gives values for the rainbow number $\mathrm{rb}([n], 3)$ determined by Riley et al. in [23] and computed by computer program.

Table 5.1 Known values for $\mathrm{rb}([n], 3)$

n	$\mathrm{rb}([n], 3)$
1	2
2	3
3	4
4	5
5	4
6	5
7	5
8	6
9	6
10	6
11	6
12	6
13	6
14	6
15	6
16	7
17	7
18	7
19	7
20	7
21	7
22	7
23	7
24	7
25	7
26	7
27	7
28	8
29	8

5.3.3 Anti-Schur Numbers of $[n]$

In this subsection some results found by Fallon, Giles, Rehm, Wagner, and Warnberg in [9] are stated about the anti-Schur number of $[n]$.

Theorem 112. *[9] For $n \geq 3$,*

$$\operatorname{as}([n], 3) = \lfloor \log_2(n) \rfloor + 2.$$

Theorem 113. *[9] For $n \geq 4$,*

$$\operatorname{as}([n], 4) = \lfloor \frac{n+7}{2} \rfloor.$$

The switch from a logarithmic to linear function when considering $\operatorname{as}([n], 3)$ versus $\operatorname{as}([n], 4)$ is interesting. In [9], Fallon et al. also conjecture that $\operatorname{as}([n], k)$ is still a linear function when $k = 5$, $k = 6$ and $k = 7$.

CHAPTER 6. CONCLUSION

In this chapter concluding remarks and projects for future work are given.

6.1 Future Projects

Although there has been a lot of research done on the anti-van der Waerden number since its conception in 2013, there are many directions to continue this research.

In Chapter 2, the anti-van der Waerden number for $[n]$ and $\mathbb{Z}_n$ is determined for $k = 3$. A further extension of these results to other abelian and non-abelian groups can be explored as well as a further investigation for $k \geq 4$ as briefly mentioned in Chapter 4.

In Chapter 3, Conjecture 36, first stated in [27], is given. This conjecture suggests Theorem 34 can be improved and that there is an upper bound similar to the bound on paths in Theorem 12, which is a logarithmic function of the number of bijacent vertices contained in the tree. Theorem 59 determines that $\text{aw}(G\square H, 3) \leq 4$ for all connected graphs G and H on at least two vertices each. One interesting question that can be explored in relation to this is to determine what graphs G and H have $\text{aw}(G\square H, 3) = 3$ and what graphs G' and H' have $\text{aw}(G'\square H', 3) = 4$ and to find any characteristics of these graphs that force the anti-van der Waerden number to be either 3 or 4. One initial thought is that it may depend on whether the diameter is odd or even, since this is the case for $P_2\square P_n$, $P_3\square P_n$, and Q_n (Proposition 52, Proposition 50, Proposition 53, Proposition 55, Corollary 67, and Proposition 68). However, this trend does not necessarily continue as seen in Corollary 62, which demonstrates that for any large enough m and n the graph $P_m\square P_n$ has anti-van der Waerden number 4 regardless of the diameter of $P_m\square P_n$. Therefore, a further categorization of these graphs would be an interesting topic of research. The results of Chapter 3 are made under the definition of a k-AP given in Chapter 1. However, the definition may be changed to yield different results. In the original definition of a 3-AP, notice $\{x, y, z\}$ have the property that $y - x = z - y$

and further $z - x = 2(y - x) = 2(z - y)$. However, in a 3-AP of a graph $\{u, v, w\}$, there is no restriction on $\mathrm{d}(u, w)$. If this distance was restricted to be either $\mathrm{d}(u, v) = \mathrm{d}(v, w) = 2\,\mathrm{d}(u, w)$ it would dramatically change the results and perhaps give an anti-van der Waerden number closer to $\mathrm{aw}([n], 3)$ or $\mathrm{aw}(\mathbb{Z}_n, 3)$ instead of most graphs having $\mathrm{aw}(G, 3) = 3$. Another variation of the definition may to restrict the 3-APs so $\mathrm{d}(u, v) = \mathrm{d}(v, w) = \mathrm{d}(u, w)$.

The beginning of an exploration of k-APs with $k \geq 4$ is discussed in Chapter 4, however; there is a lot to be done in this area. Any techniques that can be found and used on larger k-APs will be extremely useful in the development of anti-van der Waerden numbers on these larger term arithmetic progressions. Conjecture 71 seeks to explore the anti-van der Waerden number of graphs with a dominating vertex and hypothesizes the answer is either k or $k + 1$. These are shown to be the correct values for $k = 3, 4, 5$ in Observation 18 and Proposition 72. However, the conjecture remains unknown for $k \geq 6$. The bounds for k-APs of graphs with diameter 2 are shown to be tight in Corollary 78. However, Conjecture 71 suggests that if the diameter 2 graph in question has a dominating vertex, the anti-van der Waerden number is at most $k + 1$. A proof of this conjecture or a counterexample of a diameter 2 graph with a dominating vertex and anti-van der Waerden number greater than $k + 1$ would be an interesting result in either case. In Chapter 4, it is also conjectured that the bounds of Theorem 84 are tight. The question that for a graph G with diameter d, can the Ramsey construction be used to find graphs G_i such that $\mathrm{aw}(G_i, k) = k + i$ where $0 \leq i \leq R_d(P_k) - k$ is posed. An answer to either of these conjectures would be an interesting result. As seen in Section 4.5, the Ramsey number of d paths can be used as an upper bound for the anti-van der Waerden number of a graph with diameter d. This relationship also works in the other direction, that is; the anti-van der Waerden number of a graph with diameter d gives a lower bound for the Ramsey number of $R_d(P_k)$. Any further connections between the fields of Ramsey numbers and anti-van der Waerden numbers would be a useful tool for both fields of study.

In Chapter 5, an initial investigation of $[m] \times [n]$ is conducted. This work could be flushed out in more detail or developed further. Proving Conjecture 93 to show that extremal colorings of the rows are unitary would be a great stride toward furthering this area of research. At the very least, it would show Conjecture 94 to be true. The concept of the anti-van der Waerden number of directed and weighted graphs are defined in Chapter 5. These topics could lead to interesting results or a paper in the future. In addition, analogous definitions could be given for hypergraphs, graphs with multi-edges, and other variations of simple graphs. There could also be an exploration of the anti-van der Waerden number of a graphs line graph to see if it has any relationship to the graphs anti-van der Waerden number. Finally in Chapter 5, anti-Schur numbers and rainbow numbers are discussed. Both of these concepts relate to the anti-van der Waerden number since the anti-van der Waerden number satisfies the equation $x_1 + x_2 = 2x_3$ for a 3-AP $\{x_1, x_3, x_2\}$. In the anti-Schur number the equation is modified to be $\text{as}(G, k)$ must satisfy the equation $\sum_{i=1}^{k-1} x_i = x_k$ where $x_j \in G$ for all $1 \leq j \leq k$. Similarly the rainbow number $\text{rb}(G, \ell)$ must satisfy the expression $x_1 + x_2 = kx_3$. In either case these variations have had some initial investigation, but have unexplored areas. A similar area of research could be developed by modifying the equation again. One possible modification is to combine anti-Schur numbers and rainbow numbers to form the equation $\sum_{i=1}^{k-1} x_i = \ell x_k$. This anti-Schur rainbow number would be given three parameters, the mathematical object G, and fixed integers k and ℓ. Another interesting variation of this research would be to look at the product instead of the sum, that is ensure that there exists elements such that $\prod_{i=1}^{k-1} x_i = \ell x_k$. There are a multitude of variations in this direction that the research could be taken.

6.2 Conclusion

This dissertation compiled the results of anti-van der Waerden numbers. The anti-van der Waerden number for $[n]$, $\mathbb{Z}_n$, and graphs has been well explored for $k = 3$ and is known for both $\mathbb{Z}_n$ (Theorem 2) and $[n]$ (Theorem 5). The anti-van der Waerden number has been determined for most graphs (Corollary 24). A further exploration of anti-van der Waerden numbers with $k \geq 4$ is given in Chapter 4. Most notably, a connection between the anti-van der Waerden number and the Ramsey number of d paths is established in Theorem 84. The bounds of this relationship have shown to be tight for the case when the diameter d is equal to 2 (Corollary 78). Finally the relationship between anti-van der Waerden numbers, anti-Schur numbers, and rainbow numbers are examined and results on these variations of anti-van der Waerden numbers are given.

REFERENCES

[1] M. Axenovich and D. Fon-Der-Flaass. (2004). On rainbow arithmetic progressions *Electronic Journal of Combinatorics*, *11*(1), 7pp.

[2] M. Axenovich and R.R. Martin. (2006). Sub-Ramsey numbers for arithmetic progressions *Graphs and Combinatorics*, *22*(3), 297–309.

[3] F.A. Behrend. (1946). On sets of integers which contain no three terms in arithmetical progression *Proceedings of the National Academy of Sciences of the USA*, *32*, 331–332.

[4] Z. Berikkyzy, A. Schulte, and M. Young. (2017). Anti-van der Waerden numbers of 3-term arithmetic progressions *Electronic Journal of Combinatorics*, *24*(2), 9pp.

[5] E. Bevilacqua, S. King, J. Kritschgau, M. Tait, S. Tebon, and M. Young. (2018). Rainbow Numbers for $x_1 + x_2 = kx_3$ in $\mathbb{Z}_n$ Preprint. https://arxiv.org/pdf/1809.04576.pdf

[6] H. Bielak. (2009). Multicolor Ramsey Numbers for some Paths and Cycles *Discussiones Mathematicae Graph Theory*, *29*, pp.209–218.

[7] S. Butler, C. Erickson, L. Hogben, K. Hogenson, L. Kramer, R.L. Kramer, J. Lin, R.R. Martin, D. Stolee, N. Warnberg, and M. Young. (2016). Rainbow arithmetic progressions *Journal of Combinatorics*, *7*(4), pp.595–626.

[8] E. Curl, M. Jacobs, A. Schulte, and M. Young. (2015). Anti-van der Waerden number of arithmetic progressions in groups Unpublished Manuscript

[9] K. Fallon, C. Giles, H. Rehm, S. Wagner, and N. Warnberg. (2018). Anti-schur numbers for $\sum_{i=1}^{k-1} x_i = x_k$ in $[n]$ Preprint. https://arxiv.org/pdf/1901.08613.pdf

[10] R.J. Faudree and R.H. Schlep. (1975). Path Ramsey Numbers in Multicolorings *Journal of Combinatorial Theory*, Series B, *19* pp.150-160.

[11] L. Gerencesíer and A. Gyárfás. On Ramsey-Type Problems *Annales Universitatis Scientiarum Budapestinesis*, E otv os Sect. Math. 10

[12] W.T. Gowers. (2001). A new proof of Szemerédi's theorem *Geometric and Functional Analysis*, *11*(3), 465–588.

[13] R.L. Graham, B.L. Rothschild, and J.H. Spencer. (1990). Ramsey Theory. Second Edition New York, John Wiley and Sons. Print.

[14] A. Gyárfás, M. Ruszinkó, G.N. Sák ozy, and E. Szemerédi. (2007). Three-Color Ramsey Numbers for Paths *Combinatorica*, *27*, 35–69.

[15] R.W. Irving. (1974). Generalised Ramsey Numbers for Small Graphs *Discrete Mathematics*, *9*, pp.251–264.

[16] V. Jungić, J. Licht (Fox), M. Mahdian, J. Nešetril, and R. Radoičić. (2003). Rainbow arithmetic progressions and anti-Ramsey results *Combinatorics, Probability and Computing*, *12*(5), 599–620.

[17] B.M. Landman and A. Robertson. (2004). Ramsey Theory on the Integers (Student Mathematical Library, V. 24) Providence: American Mathematical Society. Print.

[18] B. Lindstr om. (1983). Undecided Ramsey-Numbers for Paths *Discrete Mathematics*, *43*, pp.111–112.

[19] J.J. OConnor and E.F. Robertson. (2013). Frank Plumpton Ramsey *The Mac Tutor History of Mathematics University of St Andrews School of Mathematics and Statistics*.

[20] J.R. Omidi and G. Raeisi. (2011). On Multicolor Ramsey Number of Paths versus Cycles *Electronic Journal of Combinatorics*, P24, *18*(1), 16pp.

[21] S. Radziszowski. (2017). Small Ramsey Numbers *Electronic Journal of Combinatorics*, *1*(15), 104pp.

[22] H. Rehm, A. Schulte, and N. Warnberg. (2018). Anti-van der Waerden numbers of Graph Products *Australasian Journal of Combinatorics*, *73*(3), 486–500.

[23] M. Riley, A. Schulte, C. Weida, and M. Young. (2019). Rainbow Numbers of $[n]$ Unpublished Manuscript

[24] V. Rosta. (2004). Ramsey Theory Applications *Electronic Journal of Combinatorics*, *DS13*, 43pp.

[25] G.N. Sárk ozy. (2016). On the Multi-Colored Ramsey Numbers of Paths and Even Cycles *Electronic Journal of Combinatorics*, # P3.53, *3*(23), 9 pages.

[26] A. Schulte. (2016). Unitary anti-van der Waerden number of arithmetic progressions Master's Thesis Iowa State University.

[27] A. Schulte, N. Warnberg, and M. Young. (Preprint). Anti-van der Waerden numbers on Graphs Preprint. https://arxiv.org/abs/1802.01509

[28] K. Uherka. (2013). An introduction to Ramsey theory and anti-Ramsey theory on the integers Master's Creative Component Iowa State University.

[29] B.L. Van der Waerden. (1927). Nieuw Archief voor Wiskunde 15212-16 *Beweis einer Baudetschen Vermutung*

[30] W.D. Wallis. (1981). On a Ramsey Number for Paths *Journal of Combinatorics, Information, & System Sciences*, *6*, pp.295–296.

[31] M. Young. (2018). Rainbow Arithmetic Progressions in Finite Abelian Groups *Journal of Combinatorics*, *9*(4), 619–629.